Estimates on the dimension of self-similar measures with overlaps

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Outline

1. Background on the dimension of self-similar measures
2. Projection entropy and lower bound estimate
3. Lower bound on the dimension of Bernoulli convolutions
4. Upper bound on the dimension of Bernoulli convolutions

Consider an IFS $\{S_i\}_{i=1}^{\ell}$ of contracting maps on $\mathbb{R}^d$. The attractor K of this IFS, is the unique nonempty compact set satisfying

$$K = \bigcup_{i=1}^{\ell} S_i(K).$$

We call K a self-similar set if S_i are similarities, that is

$$S_i(x) = \rho_i O_i x + a_i, \qquad i = 1, \ldots, \ell,$$

where O_i are orthogonal matrices, $0 < \rho_i < 1$, and $a_i \in \mathbb{R}^d$.
We call K a self-affine set if S_i are affine maps.

In the following, we assume that S_i are similarities.

Self-similar measures

Given a probability weight $\mathbf{p} = (p_1, \ldots, p_\ell)$ with $p_i > 0$ for $i = 1, \ldots, \ell$, there is a unique Borel probability measure μ supported on K such that

$$\mu = \sum_{i=1}^{\ell} p_i \, \mu \circ S_i^{-1},$$

which is called the self-similar measure associated with $\{S_i\}_{i=1}^{\ell}$ and $\mathbf{p}$.

Main question

How to determine the Hausdorff dimension of self-similar measures $\dim_H \mu$?

Progress

(Hutchinson, 1981) Suppose that $\{S_i\}_{i=1}^{\ell}$ satisfies the open set condition (OSC). Then

$$\dim_H \mu = \dim_S \mu := \frac{\sum_i p_i \log p_i}{\sum_i p_i \log \rho_i}.$$

Consider a self-similar IFS $\{S_i\}_{i=1}^{\ell}$ on $\mathbb{R}$, that is

$$S_i(x) = \rho_i x + a_i, \qquad i = 1, \ldots, \ell,$$

where $0 < \rho_i < 1$ and $a_i \in \mathbb{R}$.

- (Hochman, 2014) Suppose $\{S_i\}$ satisfies the exponential separation condition ($\exists\, c > 0$ such that $|S_{i_1 \ldots i_n} - S_{j_1 \ldots j_n}| > c^n$ for $i_1 \ldots i_n \neq j_1 \ldots j_n$). Then

  $$\dim_H \mu = \min\{1, \dim_S \mu\}.$$

  This holds if all ρ_i, a_i are algebraic numbers and no exact-overlap ($S_{i_1 \ldots i_n} \neq S_{j_1 \ldots j_n}$ for $i_1 \ldots i_n \neq j_1 \ldots j_n$).

- (Hochman, 2017) Higher dimensional extensions.
- (Rapaport, 2020) Suppose all ρ_i are algebraic numbers (no assumptions on a_i) and no exact-overlap. Then

$$\dim_H \mu = \min\{1, \dim_S \mu\}. \tag{1}$$

- (Varjú, 2019; Rapaport and Varjú, 2020) Equation (1) holds if $\rho_i = \rho_j$ for $1 \leq i < j \leq \ell$ and all a_i are rational numbers and no exact-overlap.

The main question has not yet been completely solved, especially without separation conditions.

Main goal

To provide some algorithms for numerically estimating the dimension of overlapping self-similar measures.

Notation

Let $\{S_i\}_{i=1}^{\ell}$ be an IFS on $\mathbb{R}^d$. Let (Σ, σ) be the one-sided full shift over the alphabet $\{1, \ldots, \ell\}$. Let $\pi\colon \Sigma = \{1, \ldots, \ell\}^{\mathbb{N}} \to \mathbb{R}^d$ be the coding map defined by

$$\pi(x) = \lim_{n \to \infty} S_{x_1} \circ \cdots \circ S_{x_n}(0), \qquad x = (x_n)_{n=1}^{\infty}.$$

Given a probability vector $(p_1, \ldots, p_\ell)$, let $m = \prod_{n=1}^{\infty}(p_1, \ldots, p_\ell)$ be the product measure on Σ. The push-forward $\mu = m \circ \pi^{-1}$ is the stationary measure satisfying

$$\mu = \sum_{i=1}^{\ell} p_i \, \mu \circ S_i^{-1}.$$

Projection entropy and dimension formula

Let $\mathcal{P} = \{[i] : \ i = 1, \ldots, \ell\}$ be the partition of Σ consisting of the 1st order cylinders.

The projection entropy of m under π is defined by

$$h_\pi(m) := H_m(\mathcal{P}) - H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)).$$

where $H_m(\mathcal{P}) = -\sum_i p_i \log p_i$ and $H_m(\cdot|\cdot)$ stands for the conditional entropy (see e.g., Parry (1981); Walters (1982)).

Theorem (Feng and Hu (2009))

Suppose that $\{S_i\}$ is a self-similar IFS and $\mu = m \circ \pi^{-1}$. Then μ is exact-dimensional ($\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = const$ for μ-a.e. x) and

$$\dim_H \mu = \frac{h_\pi(m)}{\sum_i p_i \log(1/\rho_i)}.$$

Upper bounds on $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d))$

Lemma

- *Let $\mathcal{D}$ be any finite Borel partition of $\mathbb{R}^d$ (or K). Then*

$$H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) \le H_m(\mathcal{P}|\pi^{-1}\mathcal{D}).$$

- $H_m(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^d)) = \lim_{\mathrm{diam}(\mathcal{D})\to 0} H_m(\mathcal{P}|\pi^{-1}\mathcal{D})$.

Lemma

For a finite Borel partition $\mathcal{D}$ of $\mathbb{R}^d$ or K, suppose that

$$\mu_i(D) := p_i\mu(S_i^{-1}D) \le y_i(D), \quad i = 1, \dots, \ell,\ D \in \mathcal{D}.$$

Then

$$\begin{aligned} H_m(\mathcal{P}|\pi^{-1}(\mathcal{B}(\mathbb{R}^d))) &\le H_m(\mathcal{P}|\pi^{-1}\mathcal{D}) \\ &= \sum_{D\in\mathcal{D}} f(\mu_1(D), \dots, \mu_\ell(D)) \\ &\le \sum_{D\in\mathcal{D}} f(y_1(D), \dots, y_\ell(D)) \end{aligned}$$

where $f(x_1, \dots, x_\ell) := (x_1 + \cdots + x_\ell) \sum_{i=1}^{\ell} \phi\left(\frac{x_i}{x_1+\cdots+x_\ell}\right)$ *and* $\phi(x) = -x \log x$.

An algorithm for measure estimation

Problem

How to estimate $\mu(A)$ from above with small error for a given Borel set $A \subset \mathbb{R}^d$?

By using self-similarity repeatedly,

$$\begin{aligned}\mu(A) &= \sum_{i=1}^{\ell} p_i \; \mu(S_i^{-1}A) \\ &= \sum_{i_1,\ldots,i_n} p_{i_1} \cdots p_{i_n} \mu(S_{i_n}^{-1} \circ \cdots \circ S_{i_1}^{-1} A) \\ &= \sum_{\substack{i_1 \ldots i_n \\ (S_{i_1 \ldots i_n}^{-1} A) \cap K \neq \emptyset}} p_{i_1 \ldots i_n} \mu(S_{i_1 \ldots i_n}^{-1} A)\end{aligned}$$

During this process, some of the expanded sets fully cover $\operatorname{supp} \mu$ and some others are disjoint from $\operatorname{supp} \mu$.

A diagram to illustrate the algorithm

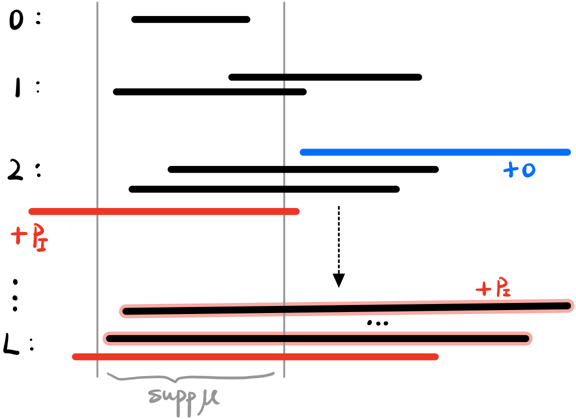


Figure: Abstract diagram of the algorithm[1]

We call the preset number L of iterations the iteration time.

[1] This algorithm is rigorously stated in the paper.

Applications to Bernoulli convolutions

For $\beta \in (1,2)$, the self-similar measure μ_β associated to the IFS

$$\{S_1(x) = x/\beta,\ S_2(x) = x/\beta + 1\}$$

and the weight $(1/2, 1/2)$ is called the Bernoulli convolution associated with β.

- $\dim_H \mu_\beta < 1$ if β is a Pisot number (Garsia, 1963).
- $\dim_H \mu_\beta = 1$ if β is algebraic but not a root of $(0, \pm 1)$ polynomials (Hochman, 2014).
- $\dim_H \mu_\beta = 1$ for some roots of $(0, \pm 1)$ polynomials (Breuillard and Varjú, 2020; Akiyama, Feng, Kempton, and Persson, 2020).
- $\dim_H \mu_\beta = 1$ if β is transcendental (Varjú, 2019).

Open question

Are Pisot numbers the only ones so that $\dim_H \mu_\beta < 1$?

Results about uniform lower bounds of $\dim_H \mu_\beta$

Theorem (Hare and Sidorov (2018))

$\dim_H \mu_\beta \geq 0.82$ *for all* $\beta \in (1, 2)$.

Theorem (Kleptsyn, Pollicott, and Vytnova (2022))

$\dim_H \mu_\beta \geq 0.96399$ *for all* $\beta \in (1, 2)$.

(Through a different approach by estimating the L^2-dimension of Bernoulli convolutions)

An improved uniform lower bound of $\dim_H \mu_\beta$

Let $\beta_3 \approx 1.839286755214...$ be the tribonacci number (i.e. the largest root of $x^3 - x^2 - x - 1 = 0$). It is known that

$$\dim_H \mu_{\beta_3} \approx 0.98040931953 \pm 10^{-11},$$

(see, Grabner-Kirschenhofer-Tichy (2002), Feng (2005)).

Theorem

For all $\beta \in (1,2)$,

$$\dim_H \mu_\beta \geq 0.98040856.$$

Moreover, the infimum $\inf_{\beta \in (1,2)} \dim_H \mu_\beta$ is attained at a parameter β_* in a small interval

$$(\beta_3 - 10^{-8}, \beta_3 + 10^{-8}).$$

Conjecture

$\dim_H \mu_\beta > \dim_H \mu_{\beta_3}$ for all $\beta \in (1,2) \setminus \{\beta_3\}$.

Strategy: reduce the parameter interval

- $\dim_H \mu_\beta \geq \dim_H \mu_{\beta^k}$ for each $\beta > 1$ and $k \in \mathbb{N}$. $[\sqrt{2}, 2]$
- $\dim_H \mu_\beta \geq 0.98041 > \dim_H \mu_{\beta_3}$ for $\beta \in [\sqrt{2}, 1.424041]$ since $\beta^2 \geq 2$ and $\dim_H \mu_\beta \geq \dim_H \mu_{\beta^2} = \log 2/(2 \log \beta)$.
  $[1.42404, 2]$

Strategy: focus on the small parameter intervals

- Take $\beta \in [\sqrt{2}, 2)$ and a small $\delta > 0$, let

$$\epsilon := \epsilon(\beta, \delta) = \begin{cases} \frac{\delta}{\beta}(1 + \frac{3}{\beta^4}) & \text{if } \beta \leq 1.5 \\ \frac{\delta}{\beta}(1 + \frac{2}{\beta^3}) & \text{if } \beta > 1.5 \end{cases}$$

- Choose an integer N, let $\mathcal{D}_{N,\beta}$ be the partition of $[0, 1]$ generated by the endpoints of $S_{I,\beta}([0, 1])$, $I \in \{1, 2\}^N$.
- Set

$$t_N = \sum_{[a,b] \in \mathcal{D}_{N,\beta}} f\left(\frac{1}{2}\mu_\beta(S_{1,\beta}^{-1}([a - \epsilon, b + \epsilon])), \ \frac{1}{2}\mu_\beta(S_{2,\beta}^{-1}([a - \epsilon, b + \epsilon])) \right).$$

  where $f(x_1, x_2) = (x_1 + x_2)(\phi(\frac{x_1}{x_1+x_2}) + \phi(\frac{x_2}{x_1+x_2}))$ and $\phi(x) = -x \log x$.
- Then for any $\beta' \in [\beta, \beta + \delta]$ with $\beta' \leq 2$,

$$\dim_H \mu_{\beta'} \geq \frac{\log 2 - t_N}{\log(\beta + \delta)}.$$

  Note $\mu_{\beta'}(S_{i,\beta'}^{-1}[a, b]) \leq \mu_\beta(S_{i,\beta}^{-1}[a - \epsilon, b + \epsilon])$.

Partition of parameters and implementation settings

BetaStart	BetaEnd	N	Iteration times	BetaStep δ
1.42404	1.43998	5	28	2E-05
1.44	1.45998	5	28	2E-05
1.46	1.49998	5	28	2E-05
1.5	1.68999	5	30	1E-05
1.69	1.77999	6	30	1E-05
1.78	1.799999	7	40	1E-06
1.8	1.839199	7	40	1E-06
1.8392	1.8392599	7	40	1E-07
1.83926	1.83927399	7	40	1E-08
1.839274	1.8392863	10	40	1E-08
1.83928631	1.839286579	10	40	1E-09
1.83928658	1.8392869339	13	40	1E-10
1.839286934	1.839287249	10	40	1E-09
1.83928725	1.83929899	10	40	1E-08
1.839299	1.83930999	7	40	1E-08
1.83931	1.8399999	7	40	1E-07
1.84	1.849999	5	30	1E-06
1.85	1.99999	5	30	1E-05

Table: Partition of $[1.42404, 2]$ and the corresponding N, δ and iteration times.

Numerical results

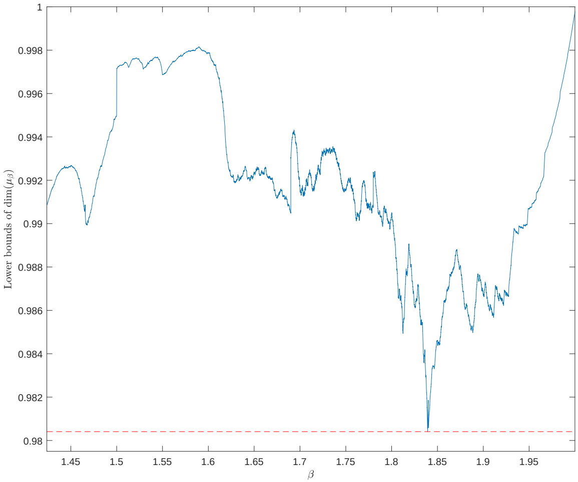


Figure: Lower bounds of $\dim_H \mu_\beta$ [2]

[2] The data is hosted at the GitHub repository zfengg/DimEstimate.

A self-affine example: setup

Consider a self-affine IFS $\{S_1, S_2\}$ on $\mathbb{R}^2$,

$$S_1(x, y) = (x/\alpha,\ y/\beta),\ S_2(x, y) = (x/\alpha + 1 - 1/\alpha,\ y/\beta + 1 - 1/\beta)$$

where $1 < \alpha < \beta < 2$, and choose a probability weight $(1/2, 1/2)$.

- Let μ be the associated self-affine measures and $m = \prod_{n=1}^{\infty}\{1/2, 1/2\}$.
- Let π denote the coding map of IFS $\{S_1, S_2\}$ and π_1 denote the coding map of IFS $\{x/\alpha, x/\alpha + 1 - 1/\alpha\}$.
- Let $\mathcal{P} = \{[1], [2]\}$ be the partition of $\{1, 2\}^{\mathbb{N}}$ according to the first symbol.

Then by (Feng and Hu, 2009, Theorem 2.11) or (Feng, 2020),

$$\begin{aligned}\dim_H \mu &= \frac{\log 2 - H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R}))}{\log \alpha} + \frac{H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R})) - H_m\left(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^2)\right)}{\log \beta} \\ &= \left(\frac{1}{\log \alpha} - \frac{1}{\log \beta}\right)\left(\log 2 - H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R})\right) + \frac{\log 2 - H_m\left(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^2)\right)}{\log \beta}.\end{aligned}$$

For each given pair (α, β), we can apply our algorithm to numerically estimate the conditional entropies $H_m(\mathcal{P}|\pi_1^{-1}\mathcal{B}(\mathbb{R}))$ and $H_m\left(\mathcal{P}|\pi^{-1}\mathcal{B}(\mathbb{R}^2)\right)$ from above, which will lead to a lower bound on $\dim_H \mu$.

A self-affine example: numerical results

(α, β)	Upper bound on $H_m(\mathcal{P}\vert\pi_1^{-1}\mathcal{B}(\mathbb{R}))$	Upper bound on $H_m(\mathcal{P}\vert\pi^{-1}\mathcal{B}(\mathbb{R}^2))$	Lower bound on $\dim_H \mu$
$(1.2, 1.4)$	0.594519457635335	0.381337271355742	1.17453512577913
$(1.3, 1.7)$	0.443677110679849	0.011983272405036	1.76440615371483
$(1.4, 1.8)$	0.358042443198116	0.000020606397791	1.60503743978513
$(1.5, 1.6)$	0.287856436058623	0.010339956295331	1.59002600229069
$(1.5, 1.7)$	0.287856436058623	0.000000000000000	1.54205227015933
$(1.7, 1.71)$	0.165391636766638	0.105399412288277	1.10640907763501
$(1.8, 1.9)$	0.064653016798699	0.035399973460099	1.03287878328719

Table: Lower bounds on the dimension of self-affine measures μ.

Upper bounds on $\dim_H \mu_\beta$ with Pisot number β

- Here we give a theoretical way to estimate the upper bound of $\dim_H \mu_\beta$ for any Pisot number β.
- Using (Feng, 2003), we built a finite tuple of non-negative matrices $\mathbf{A} = (A_1, \ldots, A_k)$ such that

$$\dim_H \mu_\beta = \frac{P'(1)}{\log(1/\beta)} = \frac{h_\eta(\sigma)}{\log \beta},$$

  where $P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|I|=n} \|A_I\|^q$, and η is the equilibrium measure corresponding to the pressure $P(q)$ at $q = 1$. Moreover, $\eta([i_1 \cdots i_n])$ can be determined from $A_{i_1} \cdots A_{i_n}$.
- Hence for any $\eta \geq 1$,

$$\begin{aligned}\dim_H \mu_\beta &\leq \frac{H_\eta(\mathcal{P}|\bigvee_{i=1}^{n} \sigma^{-i}\mathcal{P})}{\log \beta} \\ &= \frac{\sum_{I\in\Sigma_{n+1}} \phi(\eta([I])) - \sum_{J\in\Sigma_n} \phi(\eta([J]))}{\log \beta}\end{aligned}$$

  where $\phi(x) = -x\log(x)$.

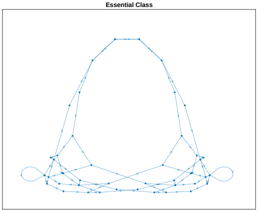
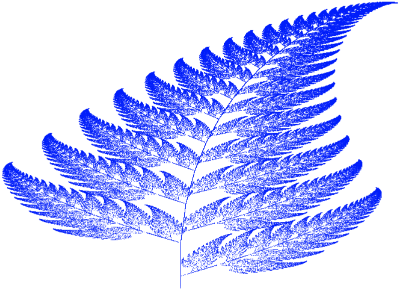


Figure: The graph for $x^3 - x^2 - 1$ (346×346 matrices $A_1, \ldots, A_{46}$)

Polynomial	numerical value of β	$\dim_H \mu_\beta$
$x^3 - x^2 - 1$	1.465571232	0.9995447 ± 10^{-7}
$x^3 - x - 1$	1.324717957	0.99999503 ± 10^{-8}
$x^3 - 2x^2 + x - 1$	1.754877666	0.9940200 ± 10^{-7}
$x^4 - 2x^3 + x - 1$	1.866760399	0.99140 ± 10^{-5}
$x^4 - x^3 - 2x^2 + 1$	1.905166167	0.98952 ± 10^{-4}

Table: Estimates on $\dim_H \mu_\beta$ for some Pisot numbers β of degree 3 or 4.



Thank you for listening!