Dimension of self-similar fractals on the line

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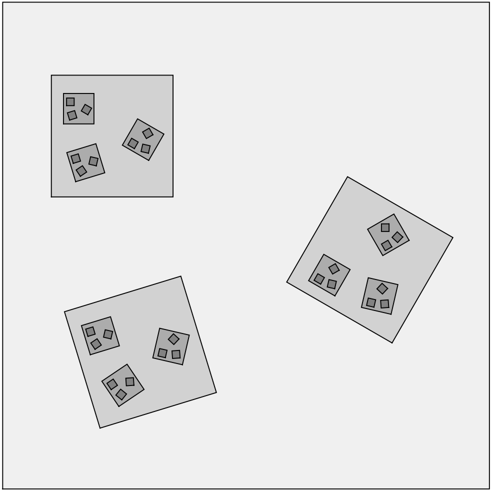
Joint work with De-Jun Feng



Self-similar sets and measures

- By iterated function system (IFS) we mean a finite family of contracting similarities on $\mathbb{R}^d$,

$$\Phi = \{\varphi_i(x) = \lambda_i O_i x + t_i\}_{i=1}^m,$$

  where O_i are orthogonal matrices, $\lambda_i \in (0,1)$ and $t_i \in \mathbb{R}^d$.

  We call Φ homogeneous if $\lambda_i O_i$ are the same for $1 \le i \le m$.

- The self-similar set K is the unique nonempty compact set so that

$$K = \bigcup_{i=1}^m \varphi_i(K).$$

- Given a probability vector $p = (p_i)_{i=1}^m$, the self-similar measure μ is the unique Borel probability measure such that

$$\mu = \sum_{i=1}^m p_i \cdot \varphi_i \mu,$$

  where $\varphi_i \mu = \mu \circ \varphi_i^{-1}$ denotes the push-forward of μ under φ_i.

Motivation, research topics and our focus

Motivation: Self-similar fractals are 'simple' and closely related to dynamical systems, number theory and harmonic analysis.

Research topics: Various properties of a self-similar measure μ.

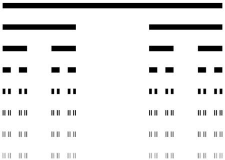
- What is the dimension of μ? Can we determine it explicitly or estimate it?
- Whether μ is absolutely continuous (to Leb.)? If so, how regular is the density?
- Does the Fourier transform of μ decay at infinity? If so, how fast is the decay?
- Projections and intersections.
- And many more...

The focus of this talk

The dimension of self-similar measures on the line.

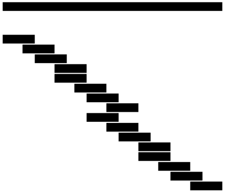
Dimension and overlaps by examples

- Cantor-Lebesgue measure ν: $\{x/3 \pm 1\}$ and $(1/2, 1/2)$.



$$\dim \nu = \frac{\log 2}{\log 3}$$

- Bernoulli convolutions μ_β for $\beta \in (1,2)$: $\{x/\beta \pm 1\}$ and $(1/2, 1/2)$.



$$\dim \mu_\beta = ?$$

Overlaps make it difficult to determine the dimension of μ:

$$\mu([x-r, x+r]) = r^{\dim \mu + o(1)} \quad \text{as } r \to 0, \text{ for } \mu\text{-a.e. } x.$$

(E.g. the length measure $\mathcal{L}$ has $\dim \mathcal{L} = 1$ since $\mathcal{L}([x-r, x+r]) = 2r^1$.)

A natural upper bound on the dimension

Let μ be a self-similar measure on $\mathbb{R}$ associated with

$$\Phi = \{\varphi_i(x) = \lambda_i x + t_i\}_{i=1}^m \quad \text{and} \quad p = (p_i)_{i=1}^m.$$

(Feng and Hu, 2009) The dimension of μ is

$$\dim \mu = \lim_{r \to 0} \frac{\log \mu([x-r, x+r])}{\log r} \quad \text{for } \mu\text{-a.e. } x. \tag{1}$$

There is a **natural upper bound:**

$$\dim \mu \le \min \left\{ 1, \frac{H(p)}{\chi} \right\}, \tag{2}$$

where $H(p) = \sum_{i=1}^m -p_i \log p_i$ is the entropy and $\chi = \sum_{i=1}^m -p_i \log|\lambda_i|$ is the Lyapunov exponent, since for μ-a.e. x, as $n \to \infty$,

$$\begin{aligned} r_n &= \exp(-n(\chi + o(1))), \\ \mu([x - r_n, x + r_n]) &\ge \exp(-n(H(p) + o(1))). \end{aligned} \tag{3}$$

The equality in (2) holds under very strong separation conditions, e.g., open set condition, because in such cases, $\ge$ can be improved to $\approx$.

Exact overlaps conjecture

How far can we relax the separation conditions to guarantee

$$\dim \mu = \min \left\{ 1, \frac{H(p)}{\chi} \right\}?$$

Definition (exact overlaps)

For an IFS $\Phi = \{\varphi_i\}_{i=1}^m$, we say Φ has exact overlaps if $\varphi_{i_1} \circ \cdots \circ \varphi_{i_n} = \varphi_{j_1} \circ \cdots \circ \varphi_{j_n}$ for some distinct words $i_1 \cdots i_n \neq j_1 \cdots j_n \in \{1, \ldots, m\}^n$.

Conjecture (exact overlaps conjecture)

If $\dim \mu < \min\{1, H(p)/\chi\}$, *then* Φ *has exact overlaps.*

A version for self-similar sets was due to (Simon, 1996).

Recent progress

Recall $\Phi = \{\lambda_i x + t_i\}_{i=1}^m$. Significant progress in recent years:

- (Hochman, 2014) There is no dimension drop if Φ is *exponentially separated*. In particular, the conjecture holds when all λ_i and t_i are algebraic numbers.
- (Shmerkin, 2019) L^q dimension version of the above.

  ($\rightarrow$ *Furstenberg intersection conjecture*)
- (Rapaport, 2022) Only require λ_i to be algebraic numbers.
- (Varjú, 2019) $\{\lambda x \pm 1\}$ and $(1/2, 1/2)$. (Bernoulli convolutions)
- (Rapaport and Varjú, 2024) $\lambda_i = \lambda$ and t_i are rational numbers. $\{\lambda x, \lambda x + 1, \lambda x + t\}$ for $(\lambda, t) \in (2^{-2/3}, 1) \times \mathbb{R}$ and equal probability weights.
- There are many other works in the theme of giving mild conditions for some stationary measure to have a explicit dimension formula.

Homogeneous IFS with algebraic translations

Theorem (Feng and F., 2024)

Let μ be the self-similar measure associated with an IFS $\Phi = \{\lambda x + t_i\}_{i=1}^m$ and a probability vector $p = (p_i)_{i=1}^m$, where t_i are algebraic numbers. If Φ has no exact overlaps, then

$$\dim \mu = \min\left\{1, \frac{\sum_{i=1}^m p_i \log p_i}{\log|\lambda|}\right\}.$$

Example

Let K be the self-similar set associated with $\left\{x/\pi + \{0, 1, \sqrt{2}\}\right\}$. Then $\dim K = \log 3/\log \pi$.

Strategy: Adapt the machinery of (Varjú, 2019) and (Rapaport and Varjú, 2024). To overcome the difficulties, we have extended some results to the setting of algebraic translations.

Heuristic proof

Fix $p = (p_i)_{i=1}^m$ and $(t_i)_{i=1}^m$ rational. ($\longrightarrow$ algebraic)

For $\eta \in (0,1)$, let μ_η be associated with $\Phi_\eta = \{\eta x + t_i\}_{i=1}^m$ and p.

Let λ be such that Φ_λ has no exact overlaps. Suppose on the contrary that $\dim \mu_\lambda < \min\{1, -H(p)/\log \lambda\}$.

1. (Breuillard and Varjú, 2019) There is an approximation (η_n) of λ:
   a. η_n is a root of a nonzero polynomial P with coefficients in $\{t_i - t_j\}_{i,j=1}^m$ and $\deg P \leq d_n$;
   b. $|\lambda - \eta_n| \leq \exp(-d_n^{1/\varepsilon})$;
   c. $\dim \mu_{\eta_n} < \dim \mu_\lambda + \varepsilon$.
2. (Breuillard and Varjú, 2020) Mahler measures of (η_n) are bounded.

By (Hochman, 2014) and a number-theoretic observation due to V. Dimitrov, the approximation (η_n) leads to a contradiction.

Find the desired approximation

Goal: A separation like n^{-Cn} for the roots of nonzero poly. in $\mathcal{P}^n$.

Here $\mathcal{P}^n = \{P \in \mathcal{D}[X] \colon \deg P \le n\}$ and $\mathcal{D} = \{t_i - t_j\}_{i,j=1}^m$.

($\mathcal{D}$ rational) Apply Mahler's result about the separation of roots of polynomials with bounded integer coefficients.

($\mathcal{D}$ algebraic)

- Find an algebraic integer θ so that $\mathcal{D} \subset \mathbb{Z}[\theta]$.

  Let $\theta_1, \dots, \theta_d$ be the algebraic conjugates of θ, and $\sigma_i \colon \mathbb{Q}(\theta) \to \mathbb{Q}(\theta_i)$, $f(\theta) \mapsto f(\theta_i)$ be the field isomorphisms.

- Consider $P \in \mathcal{D}[X]$. Note $\sigma_i(P) \in \mathbb{Z}[\theta_i][X]$. Define

$$F = \prod_{i=1}^{d} \sigma_i(P).$$

  Then $F \in \mathbb{Z}[X]$.

- Apply Mahler's result to F.

Boundedness of Mahler measure

- For an algebraic number α with minimal polynomial $a_d \prod_{i=1}^{d}(X - \alpha_i)$ in $\mathbb{Z}[X]$, the Mahler measure of α is

$$M(\alpha) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

- For $\nu = \sum_j p_j \delta_{t_j}$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the random walk entropy is

$$h_{\nu,\lambda} = \lim_{n\to\infty} \frac{H\left(\sum_{i=0}^{n-1} \xi_i \lambda^i\right)}{n},$$

where $(\xi_i)_{i=0}^{\infty}$ are i.i.d. random variables with common law ν and $H(X) = \sum_x -\mathbb{P}\{X = x\} \log \mathbb{P}\{X = x\}$ for a discrete random variable X.

Goal for bounding $M(\eta_n)$: If $M(\eta_n) \to \infty$, then $h_{\nu,\eta_n} \to H(p)$.

Given the desired algebraic approximation (η_n) of λ, there exists $\varepsilon > 0$:

$$\min\left\{1, \frac{h_{\nu,\eta_n}}{-\log \eta_n}\right\} = \dim \mu_{\eta_n} < \min\left\{1, \frac{H(p)}{-\log \lambda}\right\} - \varepsilon. \qquad (1)$$

($\mathcal{D}$ rational) $h_{\nu,\eta} \geq f_\nu(\widetilde{M}_{\mathbb{Q}}(\eta))$, where $f_\nu(A) \to H(p)$ as $A \to \infty$ and

$$\widetilde{M}_{\mathbb{K}}(\eta) = \prod_{\widetilde{\eta}\text{: conjugates of } \eta \text{ over } \mathbb{K}} \frac{1}{\min\{1, |\widetilde{\eta}|\}}.$$

Note $\widetilde{M}_{\mathbb{Q}}(\eta) \approx M(\eta)$.

($\mathcal{D}$ algebraic)

- Find some 'conjugate' system such that

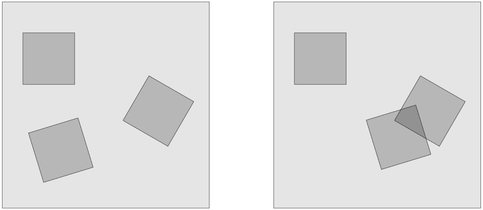
$$h_{\nu',\eta'} = h_{\nu,\eta} \quad \text{and} \quad \widetilde{M}_{\mathbb{Q}(\theta')}(\eta') \gtrsim M(\eta). \qquad (2)$$

- Since $\theta' \in \mathbb{C}$, we establish

$$h_{\nu',\eta'} \geq f_{\nu'}(\widetilde{M}_{\mathbb{Q}(\theta')}(\eta')). \qquad (3)$$

Dimension estimates for overlapping self-similar measures

If $\dim \mu < \min\{1, H(p)/\chi\}$, how to compute or estimate $\dim \mu$?



no overlaps   overlapping

- With some separation conditions, e.g., finite type condition, $\dim \mu$ is relatively well understood.
- It is challenging to determine the dimension of overlapping self-similar measures.

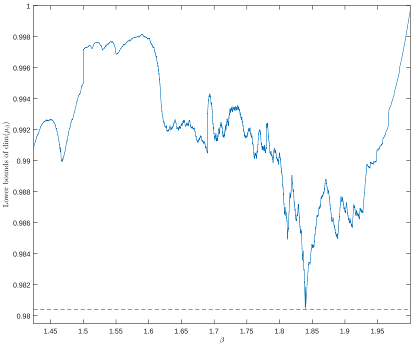
Lower bounds on $\dim \mu$

Based on the projection entropy and self-similarity, we present a method to estimate the dimension of overlapping self-similar measures from below.

Theorem (Feng and F., 2022)

For Bernoulli convolutions μ_β, $\beta \in (1,2)$, $\dim \mu_\beta \geq 0.98040856$.

- (Hare and Sidorov, 2018) 0.82.
- (Kleptsyn, Pollicott, and Vytnova, 2022) 0.96399 through a different approach.



For $\beta_3 \approx 1.839$ called the tribonacci number $(x^3 - x^2 - x - 1 = 0)$,

$$\dim \mu_{\beta_3} = 0.98040931953 \pm 10^{-11}.$$

Recall our uniform lower bound: 0.98040856.

Conjecture (Feng and F., 2022)

$\dim \mu_{\beta_3} = \inf_{\beta \in (1,2)} \dim \mu_\beta.$

Upper bounds on $\dim \mu_\beta$ when β is Pisot

- When β is a Pisot number, e.g., the golden ratio $(\sqrt{5}+1)/2$, we establish a new relation that

$$\dim \mu_\beta = \frac{h(\eta)}{\log \beta}.$$

where $h(\eta)$ is the measure-theoretic entropy of some equilibrium state η.

- Computable upper bounds for $h(\eta)$ based on products of matrices.

Example (Feng and F., 2022)

$\dim \mu_\alpha = 0.999995036 \pm 10^{-9}$, where $\alpha \approx 1.325$ is the largest root of $x^3 - x - 1 = 0$ (the smallest Pisot number).

Question (Erdős?)

Is there some $\beta \in (1, \alpha)$ so that $\dim \mu_\beta < 1$ or μ_β is singular?

Takeaways

- Self-similar sets and measures are an important family of dynamically defined fractals.
- We study various properties of them: dimension, absolute continuity, Fourier decay, projection and intersection...
- Dimension is morally the growth rate of measures of shrinking balls; Overlaps are the trouble in determining dimensions.
- Exact overlaps conjecture says that if there are no exact overlaps, then the dimension should be the expected value.
- We prove it for homogeneous IFS with algebraic translations, and present some algorithms to estimate the dimension.

Thank you for listening!

Self-similar pizza by ChatGPT