

On dimension rigidity of planar self-conformal measures

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Self-conformal fractals on $\mathbb{C}$

- *Holomorphic IFS* $\Phi = \{\varphi_i\}_{i \in \Lambda}$ on a bounded domain $\Omega \subset \mathbb{C}$: $\#\Lambda < \infty$, and each φ_i is a holomorphic injection on a domain $\Omega' \supset \Omega$ such that

$$\varphi_i(\overline{\Omega}) \subset \Omega \quad \text{and} \quad 0 < |\varphi'_i(z)| < 1 \quad \text{for } z \in \overline{\Omega}.$$

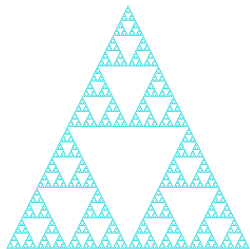
- *Self-conformal set*: the unique nonempty compact set $K \subset \Omega$ such that

$$K = \bigcup_{i \in \Lambda} \varphi_i(K).$$

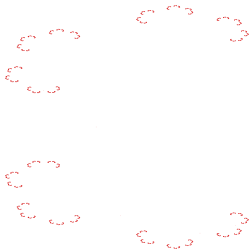
- *Self-conformal measure*: the unique Borel probability measure μ on $\mathbb{C}$ such that

$$\mu = \sum_{i \in \Lambda} p_i \cdot \varphi_i \mu,$$

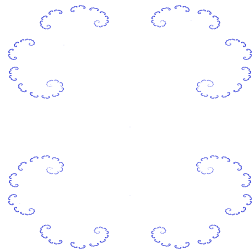
where $p = (p_i)_{i \in \Lambda}$ is a given probability vector with $p_i > 0$.



Sierpiński triangle



Limit set of Schottky semigroup



Julia set of $z \mapsto z^2 + 0.4$

For above **separated** examples, $\dim_{\mathbb{H}} K$ is well understood.

Question

How to determine **dimensions**, when Φ has **substantial overlaps**?

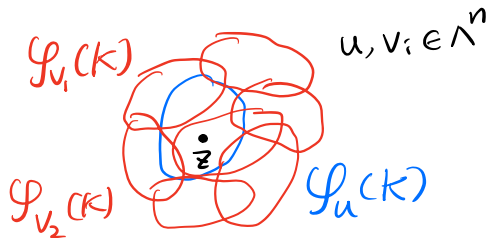
Natural upper bound

Dimension of μ : by D.-J. Feng-H. Hu ('09),

$$\dim \mu = \lim_{r \rightarrow 0} \frac{\log \mu(B(z, r))}{\log r} \quad \text{for } \mu\text{-a.e. } z.$$

Entropy of p : $H(p) = \sum_{i \in \Lambda} -p_i \log p_i$.

Lyapunov exponent of (Φ, p) : $\chi = \sum_{i \in \Lambda} p_i \int -\log |\varphi'_i(z)| \, d\mu(z)$.



$$\dim \mu \leq \frac{\log 2^{-n(H(p)+o(1))}}{\log 2^{-n(\chi+o(1))}} \rightarrow \frac{H(p)}{\chi}.$$

$$\dim \mu \leq \dim_{\text{H}} K \leq 2.$$

Natural upper bound: $\dim \mu \leq \min \left\{ \frac{H(p)}{\chi}, 2 \right\}.$

Dimension rigidity

Let μ be a self-conformal measure on $\mathbb{R}^d$. If there are **no obvious obstructions**, then $\dim \mu = \min \left\{ \frac{H(p)}{\chi}, d \right\}$.

E.g., exact overlaps, trapped in certain geometric configurations,...

Exact overlaps conjecture

Let μ be a **self-similar** measure associated to an IFS Φ on $\mathbb{R}$. If Φ **freely** generates a semigroup, then $\dim \mu = \min \left\{ \frac{H(p)}{\chi}, 1 \right\}$.

A version for self-similar **set** was due to **K. Simon ('96)**.

Progress: consider $\Phi = \{x \mapsto r_i x + t_i\}_{i \in \Lambda}$ on $\mathbb{R}$.

Hochman ('14), Varjú ('19), Rapaport ('22),

Rapaport–Varjú ('24), D.-J. Feng–F. ('25), M. Wu ('25+), ...

Progress for nonlinear cases on $\mathbb{R}$

Firstly, **Hochman–Solomyak ('17)** extended Hochman's result ('14) from self-similar IFSs to that of **Möbius transformations**.

Recently, **Ariel Rapaport ('24+)** made a breakthrough in the **real-analytic** case, where the space of actions is **infinite dimensional**!

Theorem (Ariel Rapaport '24+)

Let μ be a self-conformal measure associated to a **real-analytic** IFS Φ on a bounded interval $I \subset \mathbb{R}$. Suppose Φ does not preserve any point and is **exponentially separated**. Then $\dim \mu = \min \left\{ \frac{H(p)}{\chi}, 1 \right\}$.

Definition (Exponential separation)

We say Φ is **exponentially separated** (on I) if there exists $c > 0$ such that

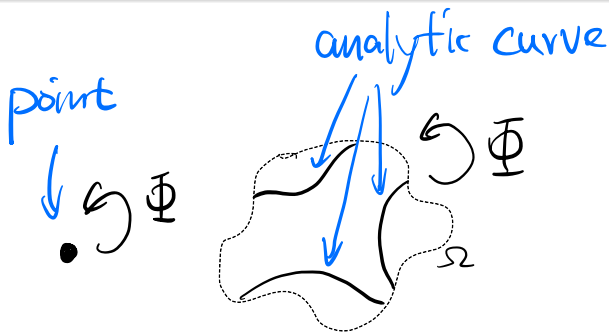
$$\|\varphi_u - \varphi_v\|_I > c^n,$$

for **distinct** $u, v \in \Lambda^n$ and **infinitely** many $n \in \mathbb{N}$, where

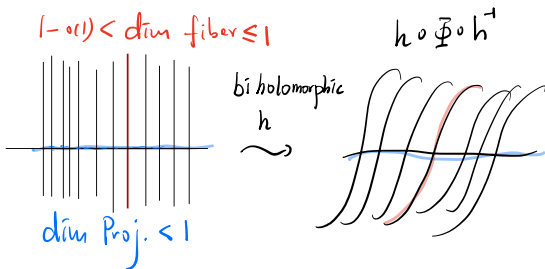
$$\|\varphi\|_I = \sup_{x \in I} |\varphi(x)| \quad \text{and} \quad \varphi_{i_1 \dots i_n} = \varphi_{i_1} \circ \dots \circ \varphi_{i_n}.$$

Additional obstructions on $\mathbb{C}$

Φ -reducibility:



Nonlinear saturation:



Recall

- Φ preserves $E \subset \mathbb{C}$ if $\varphi(E) \subset E$ for all $\varphi \in \Phi$.
- Φ is *holomorphically conjugate to a homothetic IFS* if there exists a biholomorphic map $h: \Omega \rightarrow \mathbb{C}$ such that $\{h \circ \varphi \circ h^{-1}\}_{\varphi \in \Phi}$ consists of **homothetic** maps.
- Φ is *exponentially separated* if there exists $c > 0$ such that

$$\|\varphi_u - \varphi_v\|_{\Omega} > c^n,$$

for **distinct** $u, v \in \Lambda^n$ and **infinitely** many $n \in \mathbb{N}$, where

$$\|\varphi\|_{\Omega} = \sup_{z \in \Omega} |\varphi(z)| \quad \text{and} \quad \varphi_{i_1 \dots i_n} = \varphi_{i_1} \circ \dots \circ \varphi_{i_n}.$$

Main theorem

Theorem (F.–Ariel Rapaport '26+)

Let Φ be a holomorphic IFS on a bounded domain Ω . Suppose

- ① Φ does not preserve any point or analytic curve that is closed in Ω .
- ② Φ is not holomorphically conjugate to a homothetic IFS.
- ③ Φ is exponentially separated.

Then for every positive probability vector p ,

$$\dim \mu = \min \left\{ \frac{H(p)}{\chi}, 2 \right\},$$

where μ is the self-conformal measure associated to Φ and p .

- ① is **necessary** by Φ -reducibility.
- ② is **essentially necessary** by nonlinear saturation.
- ③ is widely **conjectured** to be relaxed by Φ **freely** generating a semigroup.

Further remarks and applications

- The analogous dimension result for **overlapping** self-conformal sets.
- ① and ② $\Leftrightarrow$ the maps in Φ do not share a common fixed point, and there exists $u \in \bigcup_{n \in \mathbb{N}} \Lambda^n$ such that $\varphi'_u(z_u) \notin \mathbb{R}$, where z_u is the fixed point of φ_u .
- **Bárány–Kolossváry–Troscheit ('25+)** give **explicit** conditions for exponential separation in the **real-analytic** case.
- Expected dimension equalities hold in a nice one-parameter family of holomorphic IFSs, outside a **zero-dimensional** exceptional set (+ **Solomyak–Takahashi '22**).

About the proof

Taylor approximation technique of Rapaport ('24+) $\longrightarrow$

Notation: $\mathcal{P}_k = \{f \in \mathbb{C}[X]: \deg f \leq k\} \cong \mathbb{R}^{2(k+1)}$.

$\nu \cdot \mu$ = the pushforward of $\nu \times \mu$ under $(f, z) \mapsto f(z)$.

Theorem (Entropy increase under polynomial actions on $\mathbb{C}$)

Suppose $\dim \mu < 2$. For $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$, there exists $\delta = \delta(\varepsilon, k) > 0$ such that the following holds for $n \in \mathbb{N}$ with $\varepsilon^{-1}, k \ll n$. Let $\nu \in \mathcal{M}(\mathcal{P}_k)$ be such that $\|f\|_2 \leq 1/\varepsilon$ and $|f'(z)| \geq \varepsilon$ for $f \in \text{supp } \nu$ and $z \in \Omega$. If $\frac{1}{n}H\left(\nu, \mathcal{D}_n^{\mathcal{P}_k}\right) \geq \varepsilon$, then $\frac{1}{n}H(\nu \cdot \mu, \mathcal{D}_n^{\mathbb{C}}) \geq \dim \mu + \delta$.

Linearization method (see, e.g., Hochman–Solomyak ('17), Bárány–Hochman–Rapaport ('19)) $\longrightarrow$

Hochman's inverse theorem ('17) for translational convolutions on $\mathbb{R}^2 \cong \mathbb{C}$.

Entropy porosity: for most z and i , $\frac{1}{m}H(\mu_{z,i}, \mathcal{D}_{i+m}) < 2 - \varepsilon$.

This excludes $\left(\text{?}, \text{blue blob} \right)$

Prove **uniform entropy dimension** and assume $\dim \mu < 2$.

Entropy nonsaturation: for most z and i ,

$$\frac{1}{m}H(\mu_{z,i}, \mathcal{D}_{i+m}) - \frac{1}{m}H(P_V \mu_{z,i}, \mathcal{D}_{i+m}) < 1 - \varepsilon \quad \text{for all } V \in \mathbb{RP}^1.$$

This excludes $\left(\text{red line}, \text{blue lines} \right)$

Prove the direction cocycle $\mathbb{N} \times \Lambda^{\mathbb{N}} \ni (n, \omega) \mapsto \varphi'_{\omega|_n}(\Pi(\sigma^n \omega))\mathbb{R} \in \mathbb{RP}^1$ is not a coboundary (+ Rapaport–Ren ('25+)).

Complex-analytic inputs: rigidity and compactness of holomorphic maps...



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