Dimension of diagonal self-affine measures with overlaps

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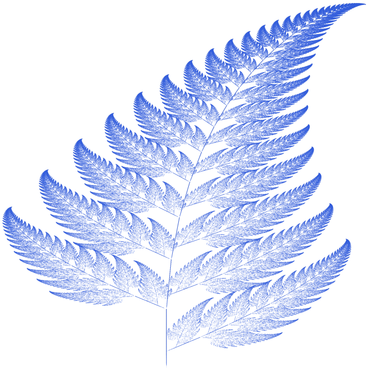
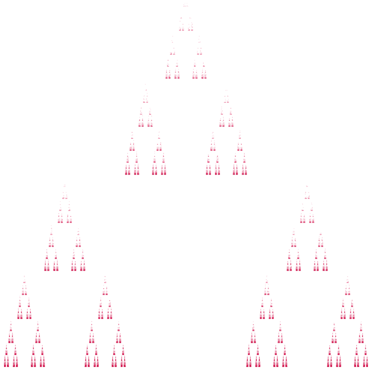
Self-affine fractals

- *Affine iterated function system (IFS):* $\Phi = \{\varphi_i\}_{i\in\Lambda}$ with $|\Lambda| < \infty$ and
$$\varphi_i(x) = A_i x + t_i \quad \text{for } x \in \mathbb{R}^d,$$
where $A_i \in \mathrm{GL}_d(\mathbb{R}), \|A_i\| < 1$ and $t_i \in \mathbb{R}^d$.
- *Self-affine/Limit set:* the unique nonempty compact set $K \subset \mathbb{R}^d$ such that
$$K = \bigcup_{i\in\Lambda} \varphi_i(K).$$
- *Self-affine/Stationary measure:* the unique Borel probability measure μ on $\mathbb{R}^d$ such that
$$\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i \mu,$$
where $p = (p_i)_{i\in\Lambda}$ is a given probability vector.
- We call K and μ *self-similar* if φ_i are similarities, i.e., $A_i \in \mathbb{R}\cdot \mathrm{O}(d)$.
- We call K and μ *diagonal* if A_i are diagonal matrices.



Barnsley fern



Hironaka curve

Question

How large are K and μ in terms of dimensions?

Which dimensions?

The Hausdorff dimension $\dim_{\mathrm{H}} K$ and the (exact) *dimension* $\dim \mu$:

$$\dim \mu = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{for } \mu\text{-a.e. } x.$$

(D.-J. Feng-Hu (2009), Bárány-Käenmäki (2017), D.-J. Feng (2023))

Natural upper bounds are defined using *singular value function*:

$$\phi^s(A) = \begin{cases} \sigma_1(A) \cdots \sigma_{\lfloor s \rfloor}(A) \sigma_{\lceil s \rceil}(A)^{s - \lfloor s \rfloor} & \text{if } 0 \le s \le d; \\ |\det A|^{s/d} & \text{if } s > d, \end{cases}$$

where $\sigma_1(A) \ge \cdots \ge \sigma_d(A)$ are singular values of $A \in \mathrm{GL}_d(\mathbb{R})$.

Affinity dimension $\dim_A \Phi$ by Falconer ('88): the unique $s \ge 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \sum_{x_1 \cdots x_n \in \Lambda^n} \phi^s(A_{x_1} \cdots A_{x_n}) = 0.$$

Lyapunov dimension $\dim_L(\Phi, p)$: the unique $t \ge 0$ such that

$$\sum_{i \in \Lambda} -p_i \log p_i + \lim_{n \to \infty} \frac{1}{n} \sum_{x_1 \cdots x_n \in \Lambda^n} p_{x_1} \cdots p_{x_n} \log \phi^t(A_{x_1} \cdots A_{x_n}) = 0.$$

Relations among dimensions

$$
\begin{array}{lccccc}
\Lambda^{\mathbb{N}} & \dim_L(\Phi, p) & \leq & \dim_A \Phi & & \\
\Big\downarrow \Pi & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\mathbb{R}^d & \dim \mu & \leq & \dim_{\mathrm{H}} K & \leq & d \\
\Big\downarrow P_V & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
V & \text{“}\dim\text{”}\, P_V\mu & \leq & \dim_{\mathrm{H}} P_V(K) & \leq & m
\end{array}
$$

- *Coding map*: $\Pi((x_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty} \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(0)$.
- $K = \Pi(\Lambda^{\mathbb{N}})$ and $\mu = \Pi\beta$, where $\beta := p^{\mathbb{N}}$ is the Bernoulli measure.
- P_V denotes orthogonal projection onto m-dim subspace $V \leq \mathbb{R}^d$.

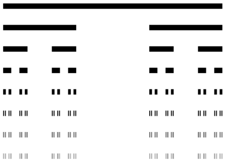
Much research aims to show “=” above across various settings.

Main difficulties: overlaps, nonconformality, saturation. . .

Overlaps by examples on the line

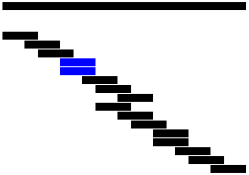
For $0 < \lambda < 1$, consider $\Phi_\lambda = \{\lambda x \pm 1\}$ on $\mathbb{R}$ and $p = (1/2, 1/2)$.

- ($\lambda = 1/3$) Cantor-Lebesgue measure ν:



$$\dim \nu = \frac{\log 2}{\log 3}$$

- ($\lambda > 1/2$) Bernoulli convolutions μ_λ:



$$\dim \mu_\lambda = ?$$

Conjecture (exact overlaps conjecture)

If Ψ on $\mathbb{R}$ generates a free semigroup, then $\dim \mu = \min\{1, \dim_L(\Psi, p)\}$.

Research target

- (Falconer '88, Solomyak '94, Jordan-Pollicott-Simon 2007) Assume $\|A_i\| < 1/2$. Then for Lebesgue almost all translations $\mathbf{t} = \{t_i\}_{i\in\Lambda}$,

$$(1)\quad \dim_{\mathrm{H}} K_{\mathbf{t}} = \min\{d, \dim_A \Phi\} \text{ and } \dim \mu_{\mathbf{t}} = \min\{d, \dim_L(\Phi, p)\}.$$

- An amount of progress in understanding typical self-affine fractals.

Reveal the generic phenomenon and exhibit new phenomenons.

Want explicit examples. The more, the better.

Research target

Find verifiable and mild conditions for (1) to hold.

Exact overlaps conjecture is within this theme.

Exponential separation

Given $\psi_1, \psi_2 \colon \mathbb{R} \to \mathbb{R}$ with $\psi_i(x) = s_i x + b_i$ for $i = 1, 2$, define

$$d(\psi_1, \psi_2) := \begin{cases} \infty & \text{if } s_1 \neq s_2; \\ |b_1 - b_2| & \text{otherwise.} \end{cases}$$

For an IFS $\Psi = \{\psi_i\}_{i \in \Lambda}$ on $\mathbb{R}$ and $n \in \mathbb{N}$, define

$$\Delta_n(\Psi) = \min\{d(\psi_u, \psi_v) \colon u \neq v \in \Lambda^n\},$$

where $\psi_{u_1 \cdots u_n} = \psi_{u_1} \circ \cdots \circ \psi_{u_n}$.

Definition (exponential separation)

We call Ψ *exponentially separated* if there exists $c > 0$ such that $\Delta_n(\Psi) > c^n$ for infinitely many $n \in \mathbb{N}$.

- If Ψ is defined by algebraic numbers and has no exact overlaps, then Ψ is exponentially separated.
- Exponential separation allows substantial overlaps and is extremely mild in the sense of small exceptions.

Self-similar:

Hochman (2014): If $d = 1$ and Φ is exponentially separated, then

(M) $$\dim \mu = \min \{d, \dim_L(\Phi, p)\}$$

(S) $$\dim K = \min \{d, \dim_A \Phi\}.$$

Towards exact overlaps conjecture: Varjú (2019), Rapaport (2022), Rapaport-Varjú (2024), D.-J. Feng-F. (2024+) ...

Hochman (2017): Self-similar fractals in $\mathbb{R}^d$.

Strongly irreducible and proximal:

For (M): Bárány-Hochman-Rapaport (2019) for $d = 2$ and SSC; Hochman-Rapaport (2022) for $d = 2$, nontrivial, and exp. sep.; Rapaport (2024) for $d = 3$ and SSC.

From (M) to (S): Morris-Shmerkin (2019) for $d = 2$, Morris-Sert (2023+) for $d \geq 3$.

Diagonal:

Various carpet-like cases: Bedford, McMullen, Przytcki-Urbanski, Gatzouras-Lalley, Kenyon-Peres, Barański, Feng-Wang, Fraser...

Some overlapping cases: Fraser-Shmerkin, Bárány-Rams-Simon...

Consider a diagonal IFS $\Phi = \{\varphi_i(x) = A_i x + t_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$:

$$A_i = \operatorname{diag}(r_{i,1}, \dots, r_{i,d}), \quad 0 < |r_{i,j}| < 1 \quad \text{and} \quad t_i = (t_{i,1}, \dots, t_{i,d}).$$

For $1 \le j \le d$, define

- *the jth Lyapunov exponent:* $\chi_j = \sum_{i\in\Lambda} -p_i \log|r_{i,j}|$.
- *the induced IFS on jth axis:* $\Phi_j = \{x \mapsto r_{i,j}x + t_{i,j}\}_{i\in\Lambda}$.

Theorem (Rapaport 2023+)

If for $1 \le j_1 < j_2 \le d$ there is $i \in \Lambda$ with $|r_{i,j_1}| \ne |r_{i,j_2}|$, and Φ_j is exponentially separated for all j, then $\dim_{\mathrm{H}} K_\Phi = \min\{d, \dim_A \Phi\}$.

Rapaport deduces it from his analogous result for diagonal self-affine measures, under the additional assumption:

$$(\bigstar) \qquad \exists\, \mathbf{c} \in \mathbb{R}^d, \forall\, i \in \Lambda\colon (\log|r_{i,1}|, \dots, \log|r_{i,d}|) \in \mathbb{R}\mathbf{c}.$$

In particular, $(\bigstar)$ holds when A_i is the same for $i \in \Lambda$.

The argument crucially relies on $(\bigstar)$. (to be discussed later)

Main result

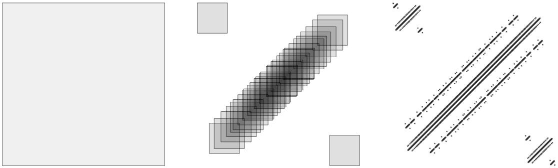
Theorem (F. 2025+)

Suppose

1. $\chi_{j_1} \neq \chi_{j_2}$ *for* $1 \leq j_1 < j_2 \leq d$.
2. Φ_j *is exponentially separated for* $1 \leq j \leq d$.

Then $\dim \mu = \min\{d, \dim_L(\Phi, p)\}$.

1. is necessary by saturation: e.g.,



2. is expected to be relaxed by no exact overlaps. When $d = 1$, it is the exact overlaps conjecture, which is famous and open.

Applications

1. Determine $\dim \mu$ when Φ is defined by algebraic parameters without exact overlaps.

2. Dimension for generic systems with explicit set of exceptions which is small in terms of packing dimension.

3. Determine the ergodic measures μ on K such that $\dim \mu = \dim_{\mathrm{H}} K$.

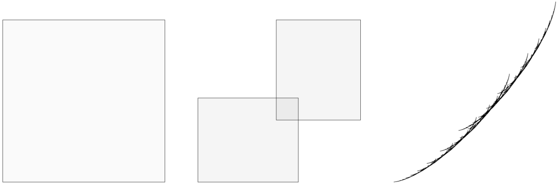
4. Dimension of orthogonal projections of certain overlapping self-affine sets.

(A concrete example is in the next page to illustrate the last two applications:)

A concrete example

Consider the IFS on $\mathbb{R}^2$:

$$\Phi = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{5}{7}x \\ \frac{3}{5}y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{3}{5}x \\ \frac{5}{7}y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$



- $\dim_A \Phi = s \approx 1.66$, where $\frac{5}{7}\left(\frac{3}{5}\right)^{s-1} + \frac{3}{5}\left(\frac{5}{7}\right)^{s-1} = 1$.
- There are exactly two ergodic measures μ_1, μ_2 which are self-affine such that

$$\dim \mu_1 = \dim \mu_2 = \dim_{\mathrm{H}} K = \dim_A \Phi.$$

- (Pyörälä (2025) + above) For all 1-dim subspace $V < \mathbb{R}^2$,

$$\dim_{\mathrm{H}} P_V(K) = 1.$$

About the proof

- Why assume condition (★)?

  To establish the uniformness of self-affine measures across scales, which is one of the key ingredients for proving entropy increase.

- The disintegration: Fix $N \in \mathbb{N}$. Partition $\Lambda^{\mathbb{N}}$ into $\mathcal{A}$ such that

$$\mathcal{A}(x) = \mathcal{A}(y) \iff A_{x_{kN+1}} \cdots A_{x_{(k+1)N}} = A_{y_{kN+1}} \cdots A_{y_{(k+1)N}} \text{ for } k \geq 0.$$

  Let $\{\beta^\omega\}_{\omega \in \Omega}$ be the disintegration of β w.r.t. $\mathcal{A}$. Then $\beta = \int \beta^\omega \, \mathrm{d}\mathbf{P}(\omega)$.
  Define $\mu^\omega = \Pi\beta^\omega$. Applying Π gives $\mu = \int \mu^\omega \, \mathrm{d}\mathbf{P}(\omega)$.

  Saglietti-Shmerkin-Solomyak (2018), Galicer-Saglietti-Shmerkin-Yavicoli (2016)

- Exact dimensionality and Ledrappier-Young formula for random measures:

  D.-J. Feng (2023) + Falconer-Jin (2014) + handle multiple disintegrations

  It also holds for partitions like $\vee_{i=0}^{\infty} \sigma^{-iN} \Gamma_N$ for any partition Γ_N of Λ^N.

- Entropy increase for random measures:

  Strategy: Rapaport (2023+)

  Difficulty: Random measures μ^ω are merely *dynamically self-affine* and nonconformal partitions $\mathcal{E}_n^\omega := A^{\omega|^n}\mathcal{D}$ depend on ω.

  Method: Feel multiscale estimates and use dynamics on $(\Omega, \mathbf{P}, T)$.

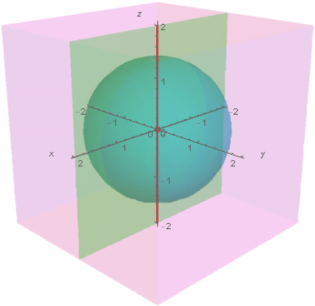
Exact dimensionality for random measures

Theorem (Exact dimensionality for disintegrations)

There exists $\dim \mathcal{A} \geq 0$ *such that for* $\mathbf{P}$*-a.e.* ω, μ^ω *is exact dimensional with dimension given by* $\dim \mathcal{A}$ *satisfying a Ledrappier-Young formula.*

$$\dim \mathcal{A} = \sum_{j=1}^{d} \frac{\mathrm{h}_{[j-1]}^{\mathcal{C},\mathcal{A}} - \mathrm{h}_{[j]}^{\mathcal{C},\mathcal{A}}}{\chi_j}.$$

For example: $0 < z\text{-axis} < xz\text{-plane} < \mathbb{R}^3$.



Theorem (Entropy increase for random measures)

Suppose $\dim \mathcal{A} < d$ *and* $\dim \pi_J \mathcal{A} = |J|$ *for each* $J \subsetneq [d]$. *For* $\varepsilon \in (0,1)$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that: Let* $\eta \in (0,1)$ *be with* $\varepsilon^{-1} \ll \eta^{-1}$. *There exists* $\Omega' \subset \Omega$ *with* $\mathbf{P}(\Omega') > 1 - \eta$ *so that for* $\omega \in \Omega'$ *and* $n \in \mathbb{N}$ *with* $\eta^{-1} \ll n$*: Let* $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ *with* $\operatorname{diam}(\operatorname{supp}\theta) \le 1/\varepsilon$ *and* $\frac{1}{n} H(\theta, \mathcal{E}_n^\omega) > \varepsilon$. *Then*

$$\frac{1}{n} H\left(\theta * \mu^\omega, \mathcal{E}_n^\omega\right) \geq \frac{1}{n} H\left(\mu^\omega, \mathcal{E}_n^\omega\right) + \delta.$$

Thank you for listening!