DIMENSION OF DIAGONAL SELF-AFFINE MEASURES WITH EXPONENTIALLY SEPARATED PROJECTIONS

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Abstract. Let μ be a self-affine measure associated with a diagonal affine iterated function system (IFS) $\Phi = \{(x_1, \ldots, x_d) \mapsto (r_{i,1}x_1 + t_{i,1}, \ldots, r_{i,d}x_d + t_{i,d})\}_{i\in\Lambda}$ on $\mathbb{R}^d$ and a probability vector $p = (p_i)_{i\in\Lambda}$. For $1 \le j \le d$, denote the j-th the Lyapunov exponent by $\chi_j := \sum_{i\in\Lambda} -p_i \log |r_{i,j}|$, and define the IFS induced by Φ on the j-th coordinate as $\Phi_j := \{x \mapsto r_{i,j}x + t_{i,j}\}_{i\in\Lambda}$. We prove that if $\chi_{j_1} \ne \chi_{j_2}$ for $1 \le j_1 < j_2 \le d$, and Φ_j is exponentially separated for $1 \le j \le d$, then the dimension of μ is the minimum of d and its Lyapunov dimension. This confirms a conjecture of Rapaport [46] by removing the additional assumption that the linear parts of the maps in Φ are contained in a 1-dimensional subgroup. One of the main ingredients of the proof involves disintegrating μ into random measures with convolution structure. In the course of the proof, we establish new results on dimension and entropy increase for these random measures.

1. Introduction

1.1. **Background and main results.** Computing the dimension of self-affine fractals remains a fundamental open problem in fractal geometry; see [7, 14]. This paper focuses on determining the dimension of diagonal self-affine measures under mild assumptions.

An affine iterated function systems (IFS) is a nonempty finite collection $\Phi = \{\varphi_i(x) = A_i x + t_i\}_{i\in\Lambda}$ of contracting affine maps on $\mathbb{R}^d$. It is well known [31] that there is a unique nonempty compact K_Φ, called the *self-affine set*, satisfying $K_\Phi = \cup_{i\in\Lambda} \varphi_i(K_\Phi)$. Given a probability vector $p = (p_i)_{i\in\Lambda}$, the associated *self-affine measure* μ is the unique Borel probability measure on $\mathbb{R}^d$ such that $\mu = \sum_{i\in\Lambda} p_i \varphi_i \mu$, where $\varphi_i \mu = \mu \circ \varphi_i^{-1}$ denotes the pushforward measure. When the linear parts $\{A_i\}_{i\in\Lambda}$ are diagonal matrices, Φ and μ are referred to as *diagonal*. In recent years, the exact dimensionality of self-affine measures has been established (see [19] for diagonal case and [4, 17] for general case). That is, there exists a number $\dim \mu$, called the *dimension* of μ, such that

$$\lim_{r\to 0} \frac{\log B(x,r)}{\log r} = \dim \mu \quad \text{for } \mu\text{-a.e. } x,$$

where $B(x,r)$ denotes the closed ball centered at x with radius r.

The dimension theory of self-affine sets and measures has been extensively studied. Notably, Falconer [13] introduced the *affinity dimension* $\dim_A \Phi$ which only depends on the linear parts $\{A_i\}_{i\in\Lambda}$. He proved that if $\|A_i\| < 1/2$ for all $i \in \Lambda$, then for Lebesgue almost all translations $\{t_i\}_{i\in\Lambda}$,

$$\dim_{\mathrm{H}} K_\Phi = \min\{d, \dim_A \Phi\}, \tag{1.1}$$

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where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. (In fact, Falconer proved this for $\|A_i\| < 1/3$; Solomyak [52] later showed that $\|A_i\| < 1/2$ suffices.) Similar results for self-affine measures were obtained by Jordan, Pollicott and Simon [32], who showed that, under the same norm condition, for Lebesgue almost all $\{t_i\}_{i\in\Lambda}$,

$$\dim \mu = \min\{d, \dim_L(\Phi, p)\}, \tag{1.2}$$

where $\dim_L(\Phi, p)$ is the *Lyapunov dimension* defined in (1.4).

While the above results provide a characterization of typical cases, finding explicit and verifiable conditions for (1.1) and (1.2) to hold remains an open challenge. Recently, significant progress has been made in this direction, particularly under the assumption that $\{A_i\}_{i\in\Lambda}$ is strongly irreducible (see [6, 29, 42] for $d = 2$ and [41, 47] for $d = 3$).

Diagonal systems, which contrast with and complement the strongly irreducible case, form a significant subclass of IFSs that have been studied since the 1980s [9, 39]. In this paper, we consider a diagonal affine IFS on $\mathbb{R}^d$:

$$\Phi = \{\varphi_i(x) = A_i x + t_i\}_{i\in\Lambda}, \tag{1.3}$$

where $A_i = \mathrm{diag}(r_{i,1}, \ldots, r_{i,d})$ $(0 < |r_{i,j}| < 1)$ are diagonal matrices, and $t_i = (t_{i,1}, \ldots, t_{i,d}) \in \mathbb{R}^d$. Let K_Φ denote the corresponding self-affine set. Given a probability vector $p = (p_i)_{i\in\Lambda}$, let μ be the self-affine measure associated with Φ and p. To state the results concerning the dimensions of K_Φ and μ, we introduce some definitions. For $1 \le j \le d$, denote the *j-th Lyapunov exponent* by $\chi_j := \sum_{i\in\Lambda} -p_i \log|r_{i,j}|$, and define the *IFS induced by Φ on the j-th coordinate* as $\Phi_j := \{x \mapsto r_{i,j}x + t_{i,j}\}_{i\in\Lambda}$. Without loss of generality, we assume after possibly permuting the coordinates that $\chi_1 \le \cdots \le \chi_d$. The *Lyapunov dimension* for Φ and p is given by

$$\dim_L(\Phi, p) = f_\Phi(H(p)), \tag{1.4}$$

where $H(p) := \sum_{i\in\Lambda} -p_i \log p_i$ is the *entropy*, and $f_\Phi\colon [0,\infty) \to [0,\infty)$ is a function defined as

$$f_\Phi(x) = \begin{cases} j + \frac{x - \sum_{b=1}^{j}\chi_b}{\chi_{j+1}} & \text{if } x \in \left[\sum_{b=1}^{j}\chi_b, \sum_{b=1}^{j+1}\chi_b\right) \text{ for some } 0 \le j \le d-1; \\ d\frac{x}{\sum_{b=1}^{d}\chi_b} & \text{if } x \in \left[\sum_{b=1}^{d}\chi_b, \infty\right). \end{cases} \tag{1.5}$$

Next, we introduce the mild separation conditions, originally arising from Hochman's seminal work [26]. Given two affine maps $\psi_1, \psi_2\colon \mathbb{R} \to \mathbb{R}$ with $\psi_i(x) = s_i x + b_i$ for $i = 1, 2$, define

$$d(\psi_1, \psi_2) := \begin{cases} \infty & \text{if } s_1 \ne s_2; \\ |b_1 - b_2| & \text{otherwise.} \end{cases}$$

For an affine IFS $\Psi = \{\psi_i\}_{i\in\Lambda}$ on $\mathbb{R}$ and $n \in \mathbb{N}$, denote $\psi_u = \psi_{u_1}\cdots\psi_{u_n}$ for $u = u_1\cdots u_n \in \Lambda^n$. Define

$$\Delta_n(\Psi) = \min\{d(\psi_u, \psi_v)\colon u, v \in \Lambda^n,\ u \ne v\} \tag{1.6}$$

and

$$S_n(\Psi) = \min\{d(\psi_u, \psi_v)\colon u, v \in \Lambda^n,\ \psi_u \ne \psi_v\}, \tag{1.7}$$

with the convention $\min\emptyset = 0$.

Definition 1.1. Let Ψ be an affine IFS on $\mathbb{R}$. We call Ψ *exponentially separated* (resp. *Diophantine*) if there exists $c > 0$ so that $\Delta_n(\Psi) > c^n$ (resp. $S_n(\Psi) > c^n$) for infinitely many $n \in \mathbb{N}$. We say Ψ has *no exact overlaps* if $\Delta_n(\Psi) > 0$ for all $n \in \mathbb{N}$, or equivalently, the semigroup generated by Ψ is free.

Remark 1.2. It follows from Definition 1.1 that Ψ is exponentially separated if and only if Ψ is both Diophantine and has no exact overlaps. Furthermore, Ψ is Diophantine if it is defined by algebraic parameters (see [26]).

Very recently, Rapaport [46] made a breakthrough in the dimension theory of diagonal self-affine sets and measures. Specifically, [46, Theorem 1.3] establishes that (1.1) holds if, for each $1 \le j_1 < j_2 \le d$ there is $i \in \Lambda$ so that $|r_{i,j_1}| \neq |r_{i,j_2}|$, and Φ_j is exponentially separated for $1 \le j \le d$. This builds on an analogous result regarding the dimension of μ ([46, Theorem 1.7]) under the additional assumption that the linear parts of Φ lie within a 1-dimensional subgroup. That is, there exist $c_1, \ldots, c_d > 0$ such that

$$(|r_{i,1}|, \ldots, |r_{i,d}|) \in \left\{(c_1^t, \ldots, c_d^t) : t \in \mathbb{R}\right\} \text{ for all } i \in \Lambda.$$

This assumption is satisfied, in particular, when A_i is the same for all $i \in \Lambda$. Regarding this assumption, Rapaport pointed out that his argument crucially depends on it, but he expects the result remains true without it (see [46, Remark 1.8]). Our main result confirms his conjecture by removing the additional assumption.

Theorem 1.3. *If $\chi_1 < \cdots < \chi_d$ and Φ_j is exponentially separated for $1 \le j \le d$, then*

$$\dim \mu = \min\{d, \dim_L(\Phi, p)\}.$$

Before discussing the proof of Theorem 1.3 in Section 1.3, we provide some remarks on the assumptions and discuss several applications.

Remark 1.4. Due to the phenomenon of saturation (see [27, Example 1.2]), it is not hard to find examples showing that the assumption $\chi_1 < \cdots < \chi_d$ cannot be dropped. For the reader's convenience, we give one such example. Let $\lambda \in \mathbb{Q} \cap (1/\sqrt{2}, 1)$ and $n > 2$ such that $\lambda^n < 1/3$. Define $\Psi = \{\psi_0(x) = \lambda x, \psi_1(x) = \lambda x + 1\}$. Consider the IFS $\Phi = \{\varphi_u\}_{u \in \{0,1\}^n}$ on $\mathbb{R}^2$ given by $\varphi_{0\cdots 0}(x, y) = (\lambda^n x + \psi_{1\cdots 1}(0), \lambda^n y)$, $\varphi_{1\cdots 1}(x, y) = (\lambda^n x, \lambda^n y + \psi_{1\cdots 1}(0))$ and $\varphi_u(x, y) = (\lambda^n x + \psi_u(0), \lambda^n y + \psi_u(0))$ for $u \notin \{0 \cdots 0, 1 \cdots 1\}$. Let μ the self-affine measure associated with Φ and the uniform probability vector p on $\{0,1\}^n$. Since the orthogonal projection of Φ onto the line $\{(t, -t) : t \in \mathbb{R}\}$ generates a Cantor set, it follows from $\lambda^n < 1/3$ and $\lambda > 1/\sqrt{2}$ that

$$\dim \mu \le 1 + \frac{\log 3}{-n \log \lambda} < 2 = \min\left\{2, \frac{\log 2}{-\log \lambda}\right\} = \min\{2, \dim_L(\Phi, p)\}.$$

On the other hand, by Remark 1.2 and $\lambda \in \mathbb{Q}$, the IFS $\Phi_1 = \Phi_2 = \Psi^n = \{\psi_u\}_{u \in \{0,1\}^n}$ is exponentially separated.

Remark 1.5. Various carpet-like examples (see e.g. [3, 9, 20, 22, 36, 39]) indicate that it is necessary to assume that Φ_j has no exact overlaps for $1 \le j \le d$. One may expect that the result remains true under this necessary assumption. Recently, Rapaport and Ren [49] verified this conjecture for homogeneous diagonal IFSs with rational translations.[1] However, even when $d = 1$, this conjecture is considered one of the major open problems in fractal geometry and well beyond our reach (see [28, 55]).

1.2. **Applications.** By Remark 1.2, the following is a direct application of Theorem 1.3.

Corollary 1.6. *Suppose $\chi_1 < \cdots < \chi_d$. If for $1 \le j \le d$, Φ_j is defined by algebraic parameters and has no exact overlaps, then* $\dim \mu = \min\{d, \dim_L(\Phi, p)\}$.

Below we determine the dimension of a concrete new example by Corollary 1.6.

Example 1.7. Let $a, b \in (1/2, 1)$ be distinct algebraic numbers such that $P(a, b) \neq 0$ for each two-variable polynomial P with coefficients in $\{0, \pm 1\}$ and $P(0, 0) = 1$. For example, choose $a = q_1/q_2, b = q_2/q_3 \in \mathbb{Q}$, where q_1, q_2, q_3 are distinct prime numbers. Let μ be the self-affine measure associated with the IFS $\Phi = \{(x, y) \mapsto (\alpha x, \beta y), (x, y) \mapsto (\beta x + 1, \alpha y + 1)\}$ on $\mathbb{R}^2$ and the probability vector $p = (p_1, 1 - p_1)$ with $p_1 \in (0, 1/2)$. Then $\dim \mu = \min\{2, \dim_L(\Phi, p)\}$.

Next, we give a result about the typical validity of (1.2) in the spirit as [26, Theorem 1.8]. By $\dim_{\mathrm{P}}$ we denote the packing dimension. Recall that $\dim_{\mathrm{H}} E \le \dim_{\mathrm{P}} E$ for $E \subset \mathbb{R}^d$. For $m \ge 2$, let Δ^{m-1} denote the set of probability vectors in $\mathbb{R}^m$.

Corollary 1.8. *Let $m \ge 2$ and let $\mathbf{t} = (t_{i,j})_{1 \le i \le m, 1 \le j \le d} \in \mathbb{R}^{dm}$ such that $t_{i_1,j} \neq t_{i_2,j}$ for $1 \le i_1 < i_2 \le m$ and $1 \le j \le d$. For $\mathbf{r} = (r_{i,j})_{1 \le i \le m, 1 \le j \le d} \in ((-1,1) \setminus \{0\})^{dm}$ and $p \in \Delta^{m-1}$, let $\mu_{\mathbf{r},p}$ denote the self-affine measure associated with the IFS $\Phi_{\mathbf{r}} = \{(x_j)_{1 \le j \le d} \mapsto (r_{i,j} x_j + t_{i,j})_{1 \le j \le d}\}_{i=1}^m$ and the probability vector p. Then, there exists $\mathcal{E}_{\mathbf{t}} \subset ((-1,1) \setminus \{0\})^{dm}$ with $\dim_{\mathrm{P}} \mathcal{E}_{\mathbf{t}} \le dm - 1$ such that for $\mathbf{r} \notin \mathcal{E}_{\mathbf{t}}$, there exists $\mathcal{F}_{\mathbf{r}} \subset \Delta^{m-1}$ with $\dim_{\mathrm{P}} \mathcal{F}_{\mathbf{r}} \le m - 2$ so that for $p \notin \mathcal{F}_{\mathbf{r}}$,* $\dim \mu_{\mathbf{r},p} = \min\{d, \dim_L(\Phi_{\mathbf{r}}, p)\}$.

Proof. For $\mathbf{r} = (r_{i,j})_{1 \le i \le m, 1 \le j \le d}$ and $1 \le j \le d$, let $\mathbf{r}_j = (r_{i,j})_{i=1}^m$. Consider the IFS $\Phi(\mathbf{r}_j) = \{x \mapsto r_{i,j} x + t_{i,j}\}_{i=1}^m$ on $\mathbb{R}$, with its coding map denoted by $\Pi_{\Phi(\mathbf{r}_j)}$ (see (1.15)). For distinct sequences $x = (x_k), y = (y_k) \in \{1, \ldots, m\}^{\mathbb{N}}$, there exists $n \in \mathbb{N}$ such that $x_n \neq y_n$ and $x_k = y_k$ for $k < n$. This gives:

$$\begin{aligned}\Delta_{x,y}(\mathbf{r}_j) &:= \Pi_{\Phi(\mathbf{r}_j)}(x) - \Pi_{\Phi(\mathbf{r}_j)}(y) \\ &= r_{x_1,j} \cdots r_{x_{n-1},j} \left((t_{x_n,j} - t_{y_n,j}) + \sum_{k=n}^{\infty} \left(r_{x_n,j} \cdots r_{x_k,j} t_{x_{k+1},j} - r_{y_n,j} \cdots r_{y_k,j} t_{y_{k+1},j} \right) \right).\end{aligned}$$

Since $t_{1,j}, \ldots, t_{m,j}$ are distinct, we have $t_{x_n} - t_{y_n} \neq 0$. Consequently, $\Delta_{x,y}(\mathbf{r}_j) \neq 0$ if the norm of $\mathbf{r}_j$ is sufficiently small, ensuring that the summation in the above expression is small, depending on $(t_{i,j})_{i=1}^m$. Thus, $\Delta_{x,y}(\mathbf{r}_j)$ is a nonzero real analytic function of $\mathbf{r}_j$ on each connected component of $((-1,1) \setminus \{0\})^m$. By applying [27, Theorem 1.10], for each $1 \le j \le d$, there exists $\mathcal{E}_j \subset ((-1,1) \setminus \{0\})^m$ with $\dim_{\mathrm{P}} \mathcal{E}_j \le m - 1$ such that $\Phi(\mathbf{r}_j)$ is exponentially separated for $\mathbf{r}_j \notin \mathcal{E}_j$. Define

$$\mathcal{E}'_{\mathbf{t}} = \bigcup_{j=1}^{d} \left\{ \mathbf{r} \in ((-1,1) \setminus \{0\})^{dm} \colon \mathbf{r}_j \in \mathcal{E}_j \right\},$$

[1] The author believes that incorporating the results from [18] into [49] can relax the assumption of rational translations to algebraic translations.

and

$$\mathcal{E} = \bigcup_{1\le j_1<j_2\le d} \left\{(r_{i,j})_{1\le i\le m,1\le j\le d} \in ((-1,1)\setminus\{0\})^{dm} \colon |r_{i,j_1}| = |r_{i,j_2}| \text{ for } 1\le i\le m\right\}.$$

Set $\mathcal{E}_{\mathbf{t}} := \mathcal{E}'_{\mathbf{t}} \cup \mathcal{E}$. Thus, for $\mathbf{r} \notin \mathcal{E}_{\mathbf{t}}$ and $1 \le j \le d$, $\Phi(\mathbf{r}_j)$ is exponentially separated. Since $\dim_{\mathrm{P}} \mathcal{E}'_{\mathbf{t}} \le dm-1$ and $\dim_{\mathrm{P}} \mathcal{E} \le dm-m$, we have $\dim_{\mathrm{P}} \mathcal{E}_{\mathbf{t}} \le dm-1$.

For $\mathbf{r} = (r_{i,j})_{1\le i\le m,1\le j\le d} \in ((-1,1)\setminus\{0\})^{dm} \setminus \mathcal{E}_{\mathbf{t}}$ and $1 \le j_1 < j_2 \le d$, define a vector $v_{j_1,j_2} := (\log|r_{i,j_1}| - \log|r_{i,j_2}|)_{i=1}^m$. Then $v_{j_1,j_2} \ne 0$ by $\mathbf{r} \notin \mathcal{E}$. If v_{j_1,j_2} is parallel to $(1,\dots,1)$, then $\Delta^{m-1} \cap v_{j_1,j_2}^{\perp} = \emptyset$, where $v_{j_1,j_2}^{\perp}$ denotes the orthogonal complement of v_{j_1,j_2}. Define

$$\mathcal{F}'_{\mathbf{r}} = \bigcup \left\{v_{j_1,j_2}^{\perp} \colon 1\le j_1<j_2\le d \text{ and } v_{j_1,j_2} \text{ is not parallel to } (1,\dots,1)\right\}.$$

Set $\mathcal{F}_{\mathbf{r}} := \Delta^{m-1} \cap \mathcal{F}'_{\mathbf{r}}$. Then $\dim_{\mathrm{P}} \mathcal{F}_{\mathbf{r}} \le m-2$. For $p \notin \mathcal{F}_{\mathbf{r}}$, the Lyapunov exponents of $\mu_{\mathbf{r},p}$ are distinct. The proof is finished by Theorem 1.3. □

We determine the measures of full dimension on certain overlapping diagonal self-affine sets (see [8, 11, 25, 33, 35, 40] for further discussion on this topic). A measure ν on K_Φ is called an *ergodic measure of full dimension* if $\dim \nu = \dim_{\mathrm{H}} K_\Phi$ and $\nu = \Pi\bar{\nu}$, where Π is the coding map in (1.15), and $\bar{\nu}$ is an ergodic shift-invariant measure on $\Lambda^{\mathbb{N}}$. Let S_d denote the symmetric group over $\{1,\dots,d\}$. For $\sigma \in S_d$, $i \in \Sigma$ and $s \ge 0$, define

$$\phi_\sigma^s(i) = \begin{cases} |r_{i,\sigma(1)}|\cdots|r_{i,\sigma(\lfloor s\rfloor)}|\cdot|r_{i,\sigma(\lfloor s\rfloor+1)}|^{s-\lfloor s\rfloor} & \text{if } s<d; \\ |r_{i,1}\cdots r_{i,d}|^{s/d} & \text{if } s\ge d. \end{cases} \tag{1.8}$$

By [23, Theorem 2.1], the affinity dimension $\dim_A \Phi$ is the unique $s \ge 0$ such that

$$\max_{\sigma\in S_d} \sum_{i\in\Lambda} \phi_\sigma^s(i) = 1. \tag{1.9}$$

Corollary 1.9. *Let Φ be as in (1.3) with $d=2$. Suppose $|r_{i,1}| \ne |r_{i,2}|$ for some $i \in \Lambda$, and Φ_1, Φ_2 are exponentially separated. Define $\Sigma := \left\{\sigma \in S_2 \colon \sum_{i\in\Lambda} \phi_\sigma^{\dim_A \Phi}(i) = 1\right\}$, which is nonempty by (1.9). If $0 < \dim_A \Phi < 2$, then the ergodic measures of full dimension on K_Φ are precisely the self-affine measures associated with Φ and the probability vectors $(\phi_\sigma^{\dim_A \Phi}(i))_{i\in\Lambda}$ for $\sigma \in \Sigma$. In particular, $\Sigma = S_2$ when $(|r_{i,1}|)_{i\in\Lambda}$ is a permutation of $(|r_{i,2}|)_{i\in\Lambda}$.*

Proof. We first show that the ergodic equilibrium states for the singular value function of diagonal matrices are Bernoulli. Let ν be an ergodic shift-invariant measure on $\Lambda^{\mathbb{N}}$. The Lyapunov dimension $\dim_L \nu$ is defined as the unique $s \ge 0$ satisfying

$$h(\nu) + \lim_{n\to\infty} \frac{1}{n} \int \log \phi^s(A_{x|n}) \, \mathrm{d}\nu(x) = 0, \tag{1.10}$$

where $h(\nu)$ denotes the measure-theoretic entropy (see [56]), $A_{x|n} = A_{x_1}\cdots A_{x_n}$ for $x = (x_n) \in \Lambda^{\mathbb{N}}$, and $\phi^s(A)$ is the singular value function of A (see [13]). For $k \in \mathbb{Z}\cap[0,d]$, it is well known [13] that $\phi^k(A) = \|A^{\wedge k}\|$, where $\wedge$ denotes the exterior product, $A^{\wedge k}$ is the linear map induced by A on $\wedge^k\mathbb{R}^d$ as $A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) := (Av_1)\wedge\cdots\wedge(Av_k)$ for $v_1,\dots,v_k \in \mathbb{R}^d$, and $\|\cdot\|$ is the standard Euclidean operator norm on $\wedge^k\mathbb{R}^d$. Since $\wedge^k\mathbb{R}^d = \mathrm{span}\left\{e_{\sigma(1)}\wedge\cdots\wedge e_{\sigma(k)} \colon \sigma \in S_d\right\}$ is a finite-dimensional vector space, where $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$, we have for $k \in \mathbb{Z} \cap [0, d]$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int \log \phi^k(A_{x|n}) \,\mathrm{d}\nu(x) &= \lim_{n\to\infty} \frac{1}{n} \int \log \|A_{x|n}^{\wedge k}\| \,\mathrm{d}\nu(x) \\
&= \lim_{n\to\infty} \frac{1}{n} \int \max_{\sigma\in S_d} \log \left\| A_{x|n}^{\wedge k} \left(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}\right) \right\| \,\mathrm{d}\nu(x) \\
&= \max_{\sigma\in S_d} \sum_{i\in\Lambda} \nu([i]) \log \phi_\sigma^k(i),
\end{aligned}$$

where $[i] := \{(x_n) \in \Lambda^{\mathbb{N}} \colon x_1 = i\}$, while in the last equality we have used that $\{A_i\}_{i\in\Lambda}$ are diagonal, and ν is shift-invariant. From this and $\phi^s(A) = \left(\phi^{\lfloor s\rfloor}(A)\right)^{\lfloor s\rfloor+1-s} \left(\phi^{\lfloor s\rfloor+1}(A)\right)^{s-\lfloor s\rfloor}$ for $s \geq 0$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \int \log \phi^s(A_{x|n}) \,\mathrm{d}\nu(x) = \max_{\sigma\in S_d} \sum_{i\in\Lambda} \nu([i]) \log \phi_\sigma^s(i). \tag{1.11}$$

Let β_ν denote the Bernoulli measure on $\Lambda^{\mathbb{N}}$ with marginal $(\nu([i]))_{i\in\Lambda}$. It is well known (see e.g. [56]) that $h(\nu) \leq h(\beta_\nu)$, with equality if and only if $\nu = \beta_\nu$. From this, (1.10) and (1.11), it follows that $\dim_L \nu \leq \dim_L \beta_\nu$, with equality if and only if $\nu = \beta_\nu$. Combining this with $\dim \Pi\nu \leq \dim_L \nu \leq \dim_A \Phi$ (see [32]), [46, Theorem 1.3], and $\dim_A \Phi < 2$ yields

$$\dim \Pi\nu \leq \dim_L \nu \leq \dim_L \beta_\nu \leq \dim_A \Phi = \dim_H K_\Phi, \tag{1.12}$$

where the second inequality is strict unless $\nu = \beta_\nu$, that is, ν is Bernoulli.

Write $s_0 := \dim_A \Phi$. By Gibbs' inequality (see e.g. [56, Lemma 9.9]) and (1.9), the probability vectors $p_\sigma := (\phi_\sigma^{s_0}(i))_{i\in\Lambda}$ for $\sigma \in \Sigma$ are precisely the probability vectors $q = (q_i)_{i\in\Lambda}$ satisfying

$$\sum_{i\in\Lambda} -q_i \log q_i + \max_{\sigma\in S_d} \sum_{i\in\Lambda} q_i \log \phi_\sigma^{s_0}(i) = \max_{\sigma\in S_d} \log \sum_{i\in\Lambda} \phi_\sigma^{s_0}(i) = 0.$$

By (1.10) and (1.11), this implies that $\dim_L(\Phi, p_\sigma) = \dim_A \Phi$ for $\sigma \in \Sigma$.

Let $\sigma \in \Sigma$, and let μ_σ be the self-affine measure associated with Φ and p_σ. By (1.12) and $\dim_L(\Phi, p_\sigma) = \dim_A \Phi$, it suffices to prove that $\dim \mu_\sigma = \dim_L(\Phi, p_\sigma)$. From Theorem 1.3, it remains to verify that $\chi_{\sigma(1)}(p_\sigma) \neq \chi_{\sigma(2)}(p_\sigma)$. If there exists $\alpha > 0$ such that $|r_{i,\sigma(1)}|/|r_{i,\sigma(2)}| = \alpha$ for all $i \in \Lambda$, then $\alpha \neq 1$ since $|r_{i,\sigma(1)}| \neq |r_{i,\sigma(2)}|$ for some $i \in \Lambda$, implying $\chi_{\sigma(1)}(p_\sigma) \neq \chi_{\sigma(2)}(p_\sigma)$. Now suppose there exist some $i_1 \neq i_2 \in \Lambda$ such that $|r_{i_1,\sigma(1)}|/|r_{i_1,\sigma(2)}| \neq |r_{i_2,\sigma(1)}|/|r_{i_2,\sigma(2)}|$. Define $t := s_0$ if $s_0 \in (0, 1]$ and $t := 2 - s_0$ if $s_0 \in (1, 2)$. Then

$$\begin{aligned}
t\left(\chi_{\sigma(2)}(p_\sigma) - \chi_{\sigma(1)}(p_\sigma)\right) &= \sum_{i\in\Lambda} p_\sigma(i) \log \frac{|r_{i,\sigma(2)}|^t}{|r_{i,\sigma(1)}|^t} \\
&< \log \sum_{i\in\Lambda} p_\sigma(i) \frac{|r_{i,\sigma(2)}|^t}{|r_{i,\sigma(1)}|^t} \\
&\leq \log \sum_{i\in\Lambda} \phi_\sigma^{s_0}(i) = 0,
\end{aligned}$$

where the strict inequality is by the concavity of $\log(\cdot)$ and $|r_{i_1,\sigma(1)}|/|r_{i_1,\sigma(2)}| \neq |r_{i_2,\sigma(1)}|/|r_{i_2,\sigma(2)}|$, while the last inequality follows from $\max_{\sigma'\in S_2} \sum_{i\in\Lambda} \phi_{\sigma'}^{s_0}(i) = \sum_{i\in\Lambda} \phi_\sigma^{s_0}(i) = 1$. Since $t > 0$, we conclude that $\chi_{\sigma(1)}(p_\sigma) \neq \chi_{\sigma(2)}(p_\sigma)$, completing the proof. □

Recently, Pyörälä [45] determined the dimension of orthogonal projections of planar diagonal self-affine measures under an irrationality condition (see [10, 16, 21, 30, 44] for earlier results). Building on this, we combine [45, Theorem 1.1] with Corollary 1.9 to obtain the dimension of orthogonal projections for a class of overlapping self-affine sets.

Corollary 1.10. *Let Φ be as in (1.3) with $d = 2$. Suppose $|r_{i,1}| \neq |r_{i,2}|$ for some $i \in \Lambda$, and Φ_1, Φ_2 are exponentially separated. Suppose further that there exist $(i_1, i_2) \in \Lambda^2$ and $(j_1, j_2) \in \{1,2\}^2$ such that $\log|r_{i_1,j_1}|/\log|r_{i_2,j_2}| \notin \mathbb{Q}$. Then $\dim_{\mathrm{H}} \pi(K_\Phi) = \min\{1, \dim_A \Phi\}$ for each orthogonal projection π onto a line not parallel to the coordinate axes. For the orthogonal projection π_j onto the j-th coordinate axis with $j = 1, 2$, $\dim_{\mathrm{H}} \pi_j(K_\Phi) = \min\{1, \dim_A \Phi_j\}$.*

1.3. **About the proof.** Theorem 1.3 is reduced from Theorem 1.12 which concerns the dimension of a disintegration of the measure μ. This disintegration is defined as follows. For any partition $\mathcal{E}$ of a set X, let $\mathcal{E}(x)$ denote the unique element of $\mathcal{E}$ containing $x \in X$. Given $u = u_1 \cdots u_n \in \Lambda^n$, define $\varphi_u = \varphi_{u_1} \circ \cdots \circ \varphi_{u_n}$. Fix $N \in \mathbb{N}$. Define the partition Γ of $\Lambda^{\mathbb{N}}$ by

$$\Gamma(x) = \Gamma(y) \text{ if and only if } A_{\varphi_{x|N}} = A_{\varphi_{y|N}} \quad \text{for } x, y \in \Lambda^{\mathbb{N}}, \tag{1.13}$$

where A_ψ denotes the linear part of an affine map ψ, and $x|N$ represents the first N digits of $x \in \Lambda^{\mathbb{N}}$. Endow $\Lambda^{\mathbb{N}}$ with the product topology, and let σ be the shift map defined by $\sigma((x_k)_{k=1}^\infty) = (x_{k+1})_{k=1}^\infty$. Set $T = \sigma^N$ and $\mathcal{A} = \vee_{n=0}^\infty T^{-n}\Gamma$. Let $\{\beta_x^{\mathcal{A}}\}_{x \in \Lambda^{\mathbb{N}}}$ be the disintegration of the Bernoulli measure $\beta := p^{\mathbb{N}}$ on $\Lambda^{\mathbb{N}}$ with respect to $\mathcal{A}$; see Section 2.4 for further details. Define the quotient space $\Omega = \Lambda^{\mathbb{N}}/\mathcal{A} \cong \{1, \ldots, |\Gamma|\}^{\mathbb{N}}$, and endow it with the pushforward measure $\mathbf{P}$ of β under the natural projection $x \mapsto \mathcal{A}(x)$. For $\omega \in \Omega$, define $\beta^\omega = \beta_x^{\mathcal{A}}$ whenever $\omega = \mathcal{A}(x)$ for some $x \in \Lambda^{\mathbb{N}}$. Then

$$\beta = \int_{\Lambda^{\mathbb{N}}} \beta_x^{\mathcal{A}} \, \mathrm{d}\beta(x) = \int_\Omega \beta^\omega \, \mathrm{d}\mathbf{P}(\omega). \tag{1.14}$$

Let $\Pi \colon \Lambda^{\mathbb{N}} \to \mathbb{R}^d$ be the coding map associated with Φ, defined by,

$$\Pi(x) = \lim_{n \to \infty} \varphi_{x_1} \circ \cdots \circ \varphi_{x_n}(0) \quad \text{for } x = (x_n)_{n=1}^\infty \in \Lambda^{\mathbb{N}}. \tag{1.15}$$

It is well known that $\mu = \Pi\beta$. For $\omega \in \Omega$, define $\mu^\omega = \Pi\beta^\omega$. Applying Π to (1.14) yields the desired disintegration:

$$\mu = \int_\Omega \mu^\omega \, \mathrm{d}\mathbf{P}(\omega). \tag{1.16}$$

Recently, similar disintegration techniques have been widely applied to study various properties of self-conformal measures; see e.g. [1, 2, 24, 34, 51, 54]. Notably, Saglietti, Shmerkin and Solomyak [51] established the typical absolute continuity of self-similar measures on the line. From this, Corollary 1.8 and [53], it seems possible to show the typical absolute continuity of diagonal self-affine measures, but we do not pursue this here. The idea of disintegrating stationary measures into well-behaved random measures was introduced by Galicer, Saglietti, Shmerkin and Yavicoli [24].

While many prior works are motivated by the infinite convolution structure of random measures, our primary goal is to construct minimal cut-sets $\mathcal{U}_n$ of the finite words over Λ. These cut-sets ensure that the cylinder sets $\{\Pi([u])\}_{u \in \mathcal{U}_n}$ have comparable diameters respectively along each coordinate. Such minimal cut-sets are naturally found in conformal settings (see [26, 48]) or under the specific assumptions on the linear parts of Φ (see [46]). However, achieving this in general non-homogeneous affine settings is almost impossible. Consequently, the additional assumption is crucial in [46]. Later in this subsection, we further illustrate how the disintegration method underpins our approach.

As a starting point, we establish the exact dimensionality of μ^ω for $\mathbf{P}$-a.e. ω; see Theorem 3.2 for a detailed statement. Theorem 3.2 is a version of [19, Theorem 2.11] (see also [17, Theorem 1.4]) in the context of disintegrations.

Theorem 1.11. *There exists* $\dim \mathcal{A} \geq 0$ *such that for* $\mathbf{P}$*-a.e.* ω*,* μ^ω *is exact dimensional with dimension given by* $\dim \mathcal{A}$*. Furthermore,* $\dim \mathcal{A}$ *satisfies a Ledrappier-Young type formula* (3.4)*.*

It is well known [57] that for an exact dimensional measure θ, commonly used notions of dimension coincide. In particular, $\dim \theta = \lim_{n\to\infty} \frac{1}{n} H(\theta, \mathcal{D}_n)$, where $\mathcal{D}_n$ denotes the dyadic partition of $\mathbb{R}^d$. For the basics of entropy, please refer to Section 2.3. By (1.16) and the concavity of entropy, we obtain

$$\dim \mu = \lim_{n\to\infty} \frac{1}{n} H\left(\int \mu_\omega \, \mathrm{d}\mathbf{P}(\omega), \mathcal{D}_n\right) \geq \lim_{n\to\infty} \int \frac{1}{n} H(\mu^\omega, \mathcal{D}_n) \, \mathrm{d}\mathbf{P}(\omega) = \dim \mathcal{A}. \tag{1.17}$$

We are now ready to state the main theorem regarding the dimension of μ^ω. For $1 \leq j \leq d$, let π_j denote the orthogonal projection from $\mathbb{R}^d$ to the j-th coordinate axis. For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the partition of $\Lambda^\mathbb{N}$ such that $\mathcal{C}_n(x) = \mathcal{C}_n(y)$ if and only if $\varphi_{x|n} = \varphi_{y|n}$ for $x, y \in \Lambda^\mathbb{N}$. The conditional entropy $H(\cdot, \cdot \mid \cdot)$ is defined in (2.4).

Theorem 1.12. *Suppose* $\chi_1 < \cdots < \chi_d$*, and* Φ_j *is Diophantine and for* $1 \leq j \leq d$*. Suppose further that for* $x, y \in \Lambda^\mathbb{N}, n \in \mathbb{N}$ *and* $1 \leq j \leq d$*,* $\pi_j \varphi_{x|n} = \pi_j \varphi_{y|n}$ *implies* $\varphi_{x|n} = \varphi_{y|n}$*. Then*

$$\dim \mathcal{A} = \min\{d, f_\Phi\left(h_{RW}(\Phi, \mathcal{A})\right)\},$$

where $f_\Phi(\cdot)$ *is as defined in* (1.5)*, and*

$$h_{RW}(\Phi, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}}\right) = \inf_n \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}}\right). \tag{1.18}$$

The limit exists by subadditivity (see (3.6)*).*

Reduction of Theorem 1.3 from Theorem 1.12. Since Φ_j is exponentially separated for $1 \leq j \leq d$, the assumptions of the theorem are satisfied, and $\mathcal{C}_{nN} = \vee_{i=0}^{nN-1} \sigma^{-i}\mathcal{P}$, where $\mathcal{P}$ denotes the partition of $\Lambda^\mathbb{N}$ based on the first digit. Note that $\widehat{\mathcal{A}} = (\vee_{i=0}^{n-1} T^{-i}\widehat{\Gamma}) \vee T^{-n}\widehat{\mathcal{A}}$, and β is Bernoulli. Then

$$\begin{aligned} H\left(\beta, \vee_{i=0}^{nN-1} \sigma^{-i}\mathcal{P} \mid \widehat{\mathcal{A}}\right) &= H\left(\beta, \vee_{i=0}^{nN-1} \sigma^{-i}\mathcal{P} \mid \vee_{i=0}^{n-1} T^{-i}\Gamma\right) && \text{(by Lemma 2.1(vii))} \\ &= H\left(\beta, \vee_{i=0}^{nN-1} \sigma^{-i}\mathcal{P}\right) - H\left(\beta, \vee_{i=0}^{n-1} T^{-i}\Gamma\right) && \text{(by Lemma 2.1(v))} \\ &= (nN)H(p) - H\left(\beta, \vee_{i=0}^{n-1} T^{-i}\Gamma\right). \end{aligned}$$

Since $\{A_{\varphi_i}\}_{i\in\Lambda}$ are commutative, by (1.13) we have $H\left(\beta, \vee_{i=0}^{n-1} T^{-i}\Gamma\right) \leq n \log|\Gamma| \leq 2n|\Lambda| \log N$. From this, (1.18) and the above equation it follows that

$$|h_{RW}(\Phi, \mathcal{A}) - H(p)| \leq 2|\Lambda| \frac{\log N}{N}.$$

From this, (1.17), Theorem 1.12, and (1.4), letting $N \to \infty$ yields that

$$\dim \mu \geq \dim \mathcal{A} = \min \{d, f_\Phi(h_{RW}(\Phi, \mathcal{A}))\} \to \min \{d, \dim_L(\Phi, p)\}.$$

This completes the proof since $\dim \mu \leq \min \{d, \dim_L(\Phi, p)\}$ always holds. □

We prove Theorem 1.12 by following the approach of Rapaport [46]. The proof relies on two key ingredients: a Ledrappier-Young type formula and an entropy increase result. For the first ingredient, we establish a Ledrappier-Young type formula for certain disintegrations of self-affine measures in Theorem 3.2, a result may be of independent interest. Based on an argument inspired by ideas from [5], this formula reduces the general case to the one where the entropy increase result can be applied.

The proof of the entropy increase result involves analyzing the multi-scale entropy of repeated self-convolutions of a measure with nonnegligible entropy, as well as the component measures of μ, along certain nonconformal partitions. In [46], the assumption that the linear parts of Φ stay in a 1-dimensional subgroup is used to find minimal cut-sets $\mathcal{U}_n$, $n \in \mathbb{N}$ of Λ^* such that

$$A_{\varphi_u} \approx A_{\varphi_v} \quad \text{for } u, v \in \mathcal{U}_n, \tag{1.19}$$

where $\approx$ means being entrywise comparable. These cut-sets are essential for estimating the asymptotic entropies of components of μ within the desired error (see [46, Section 4]). For each μ^ω, there are natural partitions $\mathcal{E}_n^\omega, n \in \mathbb{N}$ (see (4.4)). Motivated by this and (1.19), we consider the random measures μ^ω and establish the entropy increase result accordingly. However, difficulties arise because μ^ω is only dynamically self-affine (see (4.3)), and the partitions $\mathcal{E}_n^\omega$ depend on ω. To address this, we utilize the dynamics on $(\Omega, \mathbf{P})$ to prove appropriate modifications of the required lemmas. Based these lemmas, it is not difficult to adapt the arguments in [46] to derive Theorem 7.1, a version of the entropy increase result for random measures.

1.4. **Structure of the article.**

In Section 2, we introduce the basics of the conditional entropies and disintegrations. Section 3 is devoted to proving the Ledrappier-Young type formula for random measures, thereby showing Theorem 1.11. In Section 4, we define the disintegrations with respect to the linear parts of the IFS. Sections 5 and 6 are prepared for the entropy increase result which itself is proved in Section 7. Finally, Theorem 1.12 is proved in Section 8.

1.5. **Acknowledgement.**

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2. Preliminaries

In this section, we introduce the necessary notations and setup, present the basics of conditional information theory, and discuss key properties of specific disintegrations.

2.1. **Notations.** Throughout this paper, the base of $\log(\cdot)$ and $\exp(\cdot)$ is 2. For $n \in \mathbb{N}$, we define $[n] = \{1, \ldots, n\}$, with convention $[0] = \emptyset$. The normalized counting measure on $[n]$ is denoted by $\#_n$, that is, $\#_n(\{k\}) = 1/n$ for $k \in [n]$. For a finite set $\mathcal{E}$, we use $\#\mathcal{E}$ or $|\mathcal{E}|$ to represent its cardinality. By $E \subsetneq F$ we mean that E is a proper subset of F.

For a metric space X, let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and $\mathcal{M}(X)$ the set of all Borel probability measures on X. By $\mathcal{M}_c(X)$ we denote the members of $\mathcal{M}(X)$ with compact support. For $\theta \in \mathcal{M}(X)$ and $E \subset X$, the restriction of θ to E is written as $\theta|_E$, and the normalized restriction is $\theta_E = \theta|_E/\theta(E)$ if $\theta(E) > 0$.

Following [46, Section 2.1], we use the convenient notation $\ll$. Given $R_1, R_2 \geq 1$, we write $R_1 \ll R_2$ to indicate that R_2 is large with respect to (w.r.t.) R_1. Similarly, given $\varepsilon_1, \varepsilon_2 \in (0,1)$, we write $R_1 \ll \varepsilon_1^{-1}$, $\varepsilon_2^{-1} \ll R_2$ and $\varepsilon_1^{-1} \ll \varepsilon_2^{-1}$ to respectively indicate ε_1 is small w.r.t. R_1, R_2 is large w.r.t. ε_2, and ε_2 is small w.r.t. ε_1. The relation $\ll$ is clearly transitive. For example, the statement "Let $m \geq 1$, $\ell \geq L(m) \geq 1$, $k \geq K(m, \ell) \geq 1$ and $\varepsilon \leq \varepsilon_0(m, \ell, k)$ be given." is equivalent to "Let $\varepsilon \in (0,1)$ and $m, \ell, k \geq 1$ be with $m \ll \ell \ll k \ll \varepsilon^{-1}$."

2.2. **The setup.** We fix a diagonal affine IFS $\Phi = \{\varphi_i(x) = A_i x + t_i\}_{i \in \Lambda}$ on $\mathbb{R}^d$, where $A_i = \operatorname{diag}(r_{i,1}, \ldots, r_{i,d})$ with $r_{i,j} \in (-1,1) \setminus \{0\}$, and $t_i = (t_{i,j})_{j=1}^d \in \mathbb{R}^d$. The associated self-affine set is K_Φ. We fix a probability vector $p = (p_i)_{i \in \Lambda}$, and μ is the corresponding self-affine measure. Let $\Pi \colon \Lambda^{\mathbb{N}} \to K_\Phi$ denote the coding map defined as in (1.15). It is well known that $\mu = \Pi\beta$, where $\beta := p^{\mathbb{N}}$ is the Bernoulli measure on $\Lambda^{\mathbb{N}}$. For $1 \leq j \leq d$, the j-th Lyapunov exponent is $\chi_j := \sum_{i \in \Lambda} -p_i \log|r_{i,j}|$. As explained in Remark 1.4, we always assume $\chi_1 < \cdots < \chi_d$. Without loss of generality, we also assume $\operatorname{diam}(K_\Phi) \leq 1$, where $\operatorname{diam}(\cdot)$ denotes the diameter in Euclidean metric.

For $i \in \Lambda$ and $j \in [d]$, define $\varphi_{i,j} \colon \mathbb{R} \to \mathbb{R}$ by $\varphi_{i,j}(x) = r_{i,j}x + t_{i,j}$. For $\emptyset \neq J \subset [d]$, the IFS induced by Φ on $\mathbb{R}^J$ is defined as

$$\Phi_J = \{\varphi_{i,J}\}_{i \in \Lambda}, \text{ where } \varphi_{i,J}\left((x_j)_{j \in J}\right) = (\varphi_{i,j}(x_j))_{j \in J} \text{ for } i \in \Lambda. \tag{2.1}$$

For $1 \leq j \leq d$, we write Φ_j in place of $\Phi_{[j]}$. It follows that $\Phi = \Phi_{[d]}$ and $\varphi_i = \varphi_{i,[d]}$ for $i \in \Lambda$.

The collection of all finite words over Λ is denoted by Λ^*, including the empty word $\varnothing$. Write $|I| := n$ if $I \in \Lambda^n$ and $|\varnothing| := 0$. For $x = (x_i)_{i=1}^\infty \in \Lambda^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $x|n = x_1 \cdots x_n$ and $x|0 = \varnothing$. For $I \in \Lambda^*$, the cylinder set is $[I] := \{x \in \Lambda^{\mathbb{N}} \colon x||I| = I\}$. For $I = i_1 \cdots i_n \in \Lambda^n$ and $1 \leq j \leq d$, define

$$\varphi_I = \varphi_{i_1} \circ \cdots \varphi_{i_n}, \quad A^I = A_{i_1} \cdots A_{i_n}, \quad A_j^I = r_{i_1,j} \cdots r_{i_n,j}, \tag{2.2}$$

and

$$\lambda_j^I := \left|A_j^I\right| \quad \text{and} \quad \chi_j^I := -\log \lambda_j^I.$$

Let $\{e_1, \ldots, e_d\}$ be the standard basis of $\mathbb{R}^d$. For $J \subset [d]$, let π_J denote the orthogonal projection onto $\operatorname{span}\{e_j\}_{j \in J}$, that is,

$$\pi_J(x) = \sum_{j \in J} \langle e_j, x \rangle e_j \quad \text{for } x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^d$. In particular, $\pi_\emptyset$ is the zero map and $\pi_{[d]}$ is the identity map on $\mathbb{R}^d$.

2.3. **Conditional expectation, information and entropy.** Let $(X, \mathcal{B}, \theta)$ be a probability space. For a sub-σ-algebra $\mathcal{F}$ of $\mathcal{B}$, the *conditional expectation* of an integrable function f given $\mathcal{F}$ is denoted by $\mathbf{E}(\theta, f \mid \mathcal{F})$. For a countable ($\mathcal{B}$-measurable) partition ξ of X, the *conditional information* of ξ given $\mathcal{F}$ is defined as

$$\mathbf{I}(\theta, \xi \mid \mathcal{F}) = \sum_{A \in \xi} -\mathbf{1}_A \log \mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F}), \tag{2.3}$$

where $\mathbf{1}_S$ denotes the indicator function of a set S. The *conditional entropy* of ξ given $\mathcal{F}$ is

$$H(\theta, \xi \mid \mathcal{F}) := \int \mathbf{I}(\theta, \xi \mid \mathcal{F}) \, \mathrm{d}\theta = \int \sum_{A \in \xi} -\mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F}) \log \mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F}) \, \mathrm{d}\theta. \tag{2.4}$$

If $\mathcal{F} = \mathcal{N}$, the trivial σ-algebra consisting of sets of θ-measure 0 or 1, the above quantities reduce to their unconditional counterparts:

$$\mathbf{I}(\theta, \xi) = \mathbf{I}(\theta, \xi \mid \mathcal{N}) \quad \text{and} \quad H(\theta, \xi) = H(\theta, \xi \mid \mathcal{N}).$$

For $S \subset \mathcal{B}$, let $\widehat{S}$ denote the σ-algebra generated by S. Given a countable partition η, we write

$$\mathbf{I}(\theta, \xi \mid \eta) = \mathbf{I}(\theta, \xi \mid \widehat{\eta}) \quad \text{and} \quad H(\theta, \xi \mid \eta) = H(\theta, \xi \mid \widehat{\eta}). \tag{2.5}$$

In this case, the conditional entropy satisfies

$$H(\theta, \xi \mid \eta) = \sum_{A \in \eta} \theta(A) \cdot H(\theta_A, \xi),$$

where $\theta_A := \theta(A)^{-1}\theta|_A$ for $A \in \eta$ with $\theta(A) > 0$.

The following lemma summarizes key identities and properties of conditional information; see [43, 56] for details. For countable partitions $\eta_1, \ldots, \eta_n$, let $\eta_1 \vee \cdots \vee \eta_n = \vee_{i=1}^n \eta_i = \{\cap_{i=1}^n A_i \colon A_i \in \eta_i, \, 1 \le i \le n\}$. For σ-algebras $\mathcal{F}_1, \mathcal{F}_2, \ldots$, let $\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots$ or $\vee_i \mathcal{F}_i$ denote the σ-algebra generated by $\cup_i \mathcal{F}_i$. Below we take the convention $0/0 = 0$.

Lemma 2.1. *Let T be a measurable map from a separable probability space $(X, \mathcal{B}, \theta)$ to another measurable space $(Y, \mathcal{B}')$. Let $A \in \mathcal{B}$. Let ξ, η, ζ be countable partitions of X, and let $\mathcal{E}$ be a countable partition of Y, such that $H(\theta, \xi), H(\theta, \eta), H(\theta, \zeta), H(T\theta, \mathcal{E}) < \infty$. Let $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2, \ldots$ be sub-σ-algebras of $\mathcal{B}$, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{B}'$. Then the following hold.*

- *(i)* $\mathbf{E}(T\theta, g \mid \mathcal{G}) \circ T = \mathbf{E}\big(\theta, g \circ T \mid T^{-1}\mathcal{G}\big)$ *for* $g \in L^1(Y, \mathcal{B}', T\theta)$.
- *(ii)* $\mathbf{I}(T\theta, \mathcal{E} \mid \mathcal{G}) \circ T = \mathbf{I}\big(\theta, T^{-1}\mathcal{E} \mid T^{-1}\mathcal{G}\big)$.
- *(iii)* $H(T\theta, \mathcal{E} \mid \mathcal{G}) = H\big(\theta, T^{-1}\mathcal{E} \mid T^{-1}\mathcal{G}\big)$.
- *(iv)* $\mathbf{I}(\theta, \xi \vee \eta \mid \mathcal{F}) = \mathbf{I}(\theta, \xi \mid \mathcal{F}) + \mathbf{I}\Big(\theta, \eta \mid \mathcal{F} \vee \widehat{\xi}\Big)$.
- *(v)* $H(\theta, \xi \vee \eta \mid \mathcal{F}) = H(\theta, \xi \mid \mathcal{F}) + H\Big(\theta, \eta \mid \mathcal{F} \vee \widehat{\xi}\Big)$.
- *(vi) If* $\theta(A \cap F_1 \cap F_2)/\theta(F_1 \cap F_2) = \theta(A \cap F_1)/\theta(F_1)$ *for* $F_1 \in \mathcal{F}, F_2 \in \mathcal{F}_2$, *then*

$$\mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F}_1 \vee \mathcal{F}_2) = \mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F}_1).$$

- *(vii) If* $\theta(A \cap F_1 \cap F_2)/\theta(F_1 \cap F_2) = \theta(A \cap F_1)/\theta(F_1)$ *for* $A \in \xi, F_1 \in \mathcal{F}, F_2 \in \mathcal{F}_2$, *then*

$$\mathbf{I}(\theta, \xi \mid \mathcal{F}_1 \vee \mathcal{F}_2) = \mathbf{I}(\theta, \xi \mid \mathcal{F}_1) \quad \textit{and} \quad H(\theta, \xi \mid \mathcal{F}_1 \vee \mathcal{F}_2) = H(\theta, \xi \mid \mathcal{F}_1).$$

(viii) *If* $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ *for* $n \in \mathbb{N}$ *and* $\mathcal{F}_n \uparrow \mathcal{F}$, *then* $\sup_n \mathbf{I}(\theta, \xi \mid \mathcal{F}_n) \in L^1(\theta)$, *and* $\mathbf{I}(\theta, \xi, \mathcal{F}_n)$ *converges* θ *a.e. and in* $L^1(\theta)$ *to* $\mathbf{I}(\theta, \xi \mid \mathcal{F})$. *In particular,* $\lim_{n\to\infty} H(\theta, \xi \mid \mathcal{F}_n) = H(\theta, \xi \mid \mathcal{F})$.

Next, we present several useful inequalities for estimating conditional entropy. For partitions ξ and η, we say η refines ξ, denoted by $\xi \prec \eta$, if each member of η is a subset of some member of ξ.

Lemma 2.2. *Let* $(X, \mathcal{B})$ *be a measurable space, and let* $\theta, \theta_1, \dots, \theta_n$ *be probability measures on* $(X, \mathcal{B})$. *Let* ξ, η *be countable partitions of* X, *and let* $\mathcal{F}_1, \mathcal{F}_2$ *be sub-*σ*-algebras of* $\mathcal{B}$. *Then the following hold.*

(i) $H(\theta, \xi) \le \log \#\{A \in \xi \colon \theta(A) > 0\}$.

(ii) *If* $\xi \prec \eta$ *and* $\mathcal{F}_1 \subset \mathcal{F}_2$, *then* $H(\theta, \xi \mid \mathcal{F}_2) \le H(\theta, \xi \mid \mathcal{F}_1) \le H(\theta, \eta \mid \mathcal{F}_1)$.

(iii) *If* $q = (q_i)_{i=1}^n$ *is a probability vector and* $\theta = \sum_{i=1}^n q_i \theta_i$, *then*

$$\sum_{i=1}^n q_i H(\theta_i, \xi \mid \eta) \le H(\theta, \xi \mid \eta) \le \sum_{i=1}^n q_i H(\theta_i, \xi \mid \eta) + H(q).$$

(iv) *Given* $C \ge 1$, *we say that* ξ *and* η *are* C*-commensurable if for each* $A \in \xi$ *and* $B \in \eta$,

$$\#\{A' \in \xi \colon A' \cap B \ne \emptyset\} \le C \quad \text{and} \quad \#\{B' \in \xi \colon B' \cap A \ne \emptyset\} \le C.$$

If ξ *and* η *are* C*-commensurable, then* $|H(\theta, \xi) - H(\theta, \eta)| \le \log C$.

2.4. **Conditional measures and some disintegrations.** We begin with a foundational result from Rohlin's theory of conditional measures; for further details, refer to [12, 50].

Theorem 2.3 (Rohlin [50])**.** *Let* X, Y *be Euclidean spaces or product spaces of countably many finite sets. Let* η *be a partition induced by a Borel measurable map* $\pi \colon X \to Y$, *that is,* $\eta = \{\pi^{-1}(y) \colon y \in Y\}$. *Let* θ *be a Borel probability measure on* X. *Then for* θ*-a.e.* x *there exists a probability measure* θ_x^η *supported on* $\eta(x)$. *These measures are uniquely determined up zero* θ*-measure by the properties: if* $A \subset X$ *is Borel measurable, then* $x \mapsto \theta_x^\eta(A)$ *is* $\widehat{\eta}$*-measurable, and* $\theta(A) = \int \theta_x^\eta(A)\,\mathrm{d}\theta(x)$. *This means* $\theta = \int \theta_x^\eta\,\mathrm{d}\theta(x)$ *in the sense that* $\int\int f(y)\,\mathrm{d}\theta_x^\eta(y)\mathrm{d}\theta(x)$ *for* $f \in L^1(X, \mathcal{B}(X), \theta)$.

The family of measures $\{\theta_x^\eta\}_{x\in X}$ is called the *system of conditional measures of* θ *associated with* η or the *disintegration of* θ *with respect to* π.

Next, we introduce certain disintegrations and present some of their properties. Fix $N \in \mathbb{N}$. Let Γ be a partition of $\Lambda^{\mathbb{N}}$ such that for $x, y \in \Lambda^{\mathbb{N}}$, $x|N = y|N$ implies $\Gamma(x) = \Gamma(y)$. Set $T = \sigma^N$ and $\mathcal{A} = \vee_{i=0}^\infty T^{-i}\Gamma$. Define the quotient space $\Omega := \Lambda^{\mathbb{N}}/\mathcal{A} \cong \Gamma^{\mathbb{N}}$. Let $\mathbf{P}$ be the Bernoulli measure on $\Omega = \Gamma^{\mathbb{N}}$ with marginal $(\beta(\omega_1))_{\omega_1\in\Gamma}$. Specifically, for $\omega_1 \cdots \omega_n \in \Gamma^n, n \ge 1$,

$$\mathbf{P}([\omega_1 \cdots \omega_n]) = \prod_{k=1}^n \beta(\omega_k) = \beta\left\{x \in \Lambda^{\mathbb{N}} \colon \mathcal{A}(x) \in [\omega_1 \cdots \omega_n]\right\}. \tag{2.6}$$

This shows that $\mathbf{P} = \beta \circ \mathcal{A}^{-1}$, that is, $\mathbf{P}$ is the pushforward of β under $\mathcal{A}$. Here, we slightly abuse the notation by using $\mathcal{A}(x)$ to denote both a set in $\Lambda^{\mathbb{N}}$ and a sequence in $\Omega = \Gamma^{\mathbb{N}}$.

For $\omega_1 \in \Gamma$, define a measure p^{ω_1} on $\Lambda^{\mathbb{N}}$ by $p^{\omega_1} := \beta_{\omega_1}$ if $\beta(\omega_1) > 0$, and let p^{ω_1} be the zero measure if $\beta(\omega_1) = 0$. For $\omega = (\omega_n)_{n=1}^{\infty} \in \Omega$, define a product measure β^{ω} on $\Lambda^{\mathbb{N}}$ via the identification $\Lambda^{\mathbb{N}} = (\Lambda^N)^{\mathbb{N}}$ as

$$\beta^{\omega}([I]) = \prod_{k=1}^{n} p^{\omega_k}([I_k]) \quad \text{for } I = I_1 \cdots I_n \in (\Lambda^N)^n, n \geq 1. \tag{2.7}$$

Then β^{ω} is supported on $\mathcal{A}(x)$ whenever $\omega = \mathcal{A}(x)$ for some $x \in \Lambda^{\mathbb{N}}$. On the other hand, let $\{\beta_x^{\mathcal{A}}\}_{x \in \Lambda^{\mathbb{N}}}$ be the disintegration of β with respect to $\mathcal{A}$. It follows from Theorem 2.3, $\widehat{\mathcal{A}} = (\vee_{i=0}^{n-1} T^{-i}\widehat{\Gamma}) \vee \widehat{\mathcal{A}}$ and Lemma 2.1(vi) that for β-a.e. x and $I = I_1 \cdots I_n \in (\Lambda^N)^n, n \geq 1$,

$$\begin{aligned}
\beta_x^{\mathcal{A}}([I]) &= \mathbf{E}\Big(\beta, \mathbf{1}_{[I]} \mid \widehat{\mathcal{A}}\Big)(x) = \mathbf{E}\big(\beta, \mathbf{1}_{[I]} \mid \vee_{i=0}^{n-1} T^{-i}\Gamma\big)(x) \\
&= \sum_{A \in \vee_{i=0}^{n-1} T^{-i}\Gamma} \mathbf{1}_A(x) \frac{\beta([I] \cap A)}{\beta(A)} \\
&= \sum_{(\omega_k)_{k=1}^{n} \in \Gamma^n} \mathbf{1}_{[\omega_1 \cdots \omega_n]}(\mathcal{A}(x)) \prod_{k=1}^{n} p^{\omega_k}([I_k]) \\
&= \beta^{\mathcal{A}(x)}([I]),
\end{aligned}$$

where the last equality is by (2.7). Hence $\beta_x^{\mathcal{A}} = \beta^{\mathcal{A}(x)}$ for β-a.e. x. Combining this, Theorem 2.3 and $\mathbf{P} = \beta \circ \mathcal{A}^{-1}$, we obtain

$$\beta = \int_{\Lambda^{\mathbb{N}}} \beta_x^{\mathcal{A}} \, \mathrm{d}\beta(x) = \int_{\Lambda^{\mathbb{N}}} \beta^{\mathcal{A}(x)} \, \mathrm{d}\beta(x) = \int_{\Omega} \beta^{\omega} \, \mathrm{d}\mathbf{P}(\omega). \tag{2.8}$$

Recall the coding map Π from (1.15). For $\omega \in \Omega$, define $\mu^{\omega} := \Pi\beta^{\omega}$. Applying Π to (2.8) yields a disintegration of μ as

$$\mu = \int_{\Omega} \mu^{\omega} \, \mathrm{d}\mathbf{P}(\omega). \tag{2.9}$$

For $\omega \in \Omega$, the random measure μ^{ω} satisfies the *dynamical self-affinity.* By abuse of notation, let T be the shift map on Ω, defined by $T((\omega_n)_{n=1}^{\infty}) = (\omega_{n+1})_{n=1}^{\infty}$. Using (2.7), we have, for $\omega \in \Omega$,

$$T\beta^{\omega} = \beta^{T\omega}, \tag{2.10}$$

and so for $u \in \Lambda^N$,

$$T(\beta^{\omega}|_{[u]}) = \beta^{\omega}([u])\beta^{T\omega}. \tag{2.11}$$

From (1.15) it follows that for $u \in \Lambda^*$,

$$\varphi_u \circ \Pi \circ \sigma^{|u|} = \Pi \quad \text{on } [u]. \tag{2.12}$$

Thus, μ^ω satisfies the dynamical self-affinity:

$$\begin{aligned}\mu^\omega = \Pi\beta^\omega &= \sum_{u\in\Lambda^N} \Pi\beta^\omega|_{[u]} \\ &= \sum_{u\in\Lambda^N} (\varphi_u \Pi T)\beta^\omega|_{[u]} && \text{(by (2.12))} \\ &= \sum_{u\in\Lambda^N} (\varphi_u\Pi)\left(\beta^\omega([u])\beta^{T\omega}\right) && \text{(by (2.11))} \\ &= \sum_{u\in\Lambda^N} \beta^\omega([u])\cdot\varphi_u\mu^{T\omega}. && \text{(by } \mu^{T\omega} = \Pi\beta^{T\omega}\text{)}\end{aligned} \tag{2.13}$$

3. Exact dimensionality for disintegrations

In this section, we establish the exact dimensionality of certain random measures and show that their dimension satisfies a Ledrappier-Young type formula. To prove these results, we adapt the approach from deterministic case of Feng [17].

For $J \subset [d]$, define the partition ξ_J of $\Lambda^{\mathbb{N}}$ as

$$\xi_J(x) = \xi_J(y) \quad \text{if and only if} \quad \pi_J\Pi(x) = \pi_J\Pi(y) \quad \text{for } x, y \in \Lambda^{\mathbb{N}}. \tag{3.1}$$

Note that $\widehat{\xi_J} = \Pi^{-1}\pi_J^{-1}\mathcal{B}(\mathbb{R}^d) \pmod 0$.

Theorem 3.1. *Let $N \in \mathbb{N}$. Let $\mathcal{C}$ be a partition of $\Lambda^{\mathbb{N}}$ such that for $x, y \in \Lambda^{\mathbb{N}}$, $\mathcal{C}(x) = \mathcal{C}(y)$ implies $\varphi_{x|N} = \varphi_{y|N}$. Let Γ be a partition of $\Lambda^{\mathbb{N}}$ such that for $x, y \in \Lambda^{\mathbb{N}}$, $x|N = y|N$ implies $\Gamma(x) = \Gamma(y)$. Set $T = \sigma^N$ and $\mathcal{A} = \vee_{i=0}^{\infty} T^{-i}\Gamma$. Let $1 \le j_1 < \cdots < j_s \le d$ and write $J = \{j_1, \ldots, j_s\}$. For $0 \le b \le s$, set $J_b = \{j_1, \ldots, j_b\}$. Then for β-a.e. y, $\beta_y^{\mathcal{A}}$-a.e. x and $0 \le k \le l \le s$, the measure $\pi_{J_l}\Pi\beta_{y,x}^{\mathcal{A},\xi_{J_k}} := \pi_{J_l}\Pi(\beta_y^{\mathcal{A}})_x^{\xi_{J_k}}$ is exact dimensional with*

$$\dim \pi_{J_l}\Pi\beta_{y,x}^{\mathcal{A},\xi_{J_k}} = \sum_{b=k+1}^{l} \frac{\mathrm{H}_{J_{b-1}}^{\mathcal{A}} - \mathrm{H}_{J_b}^{\mathcal{A}}}{\chi_{j_b}}, \tag{3.2}$$

where for $I \subset [d]$,

$$\mathrm{H}_I^{\mathcal{A}} = \frac{1}{N} H\left(\beta, \mathcal{C} \mid \widehat{\mathcal{A}} \vee \widehat{\xi_I}\right). \tag{3.3}$$

Theorem 3.1 has following consequence which is a general and detailed version of Theorem 1.11.

Theorem 3.2. *For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the partition of $\Lambda^{\mathbb{N}}$ defined by $\mathcal{C}_n(x) = \mathcal{C}_n(y)$ if and only if $\varphi_{x|n} = \varphi_{y|n}$ for $x, y \in \Lambda^{\mathbb{N}}$. Let $N \in \mathbb{N}$. Let Γ be a partition of $\Lambda^{\mathbb{N}}$ such that for $x, y \in \Lambda^{\mathbb{N}}$, $x|N = y|N$ implies $\Gamma(x) = \Gamma(y)$. Set $\mathcal{A} = \vee_{i=0}^{\infty}\sigma^{-iN}\Gamma$. Let $1 \le j_1 < \cdots < j_s \le d$ and write $J = \{j_1, \ldots, j_s\}$. For $0 \le b \le s$, set $J_b = \{j_1, \ldots, j_b\}$. Then for β-a.e. y, the measure $\pi_J\Pi\beta_y^{\mathcal{A}}$ is exact dimensional with dimension given by*

$$\dim \pi_J \mathcal{A} = \sum_{b=1}^{s} \frac{\mathrm{h}_{J_{b-1}}^{\mathcal{C},\mathcal{A}} - \mathrm{h}_{J_b}^{\mathcal{C},\mathcal{A}}}{\chi_{j_b}}, \tag{3.4}$$

where for $I \subset [d]$,

$$\mathrm{h}_I^{\mathcal{C},\mathcal{A}} = \lim_{n\to\infty} \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi_I}\right) = \inf_n \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi_I}\right), \tag{3.5}$$

and $\mathrm{h}^{\mathcal{C},\mathcal{A}}_{J_{b-1}} - \mathrm{h}^{\mathcal{C},\mathcal{A}}_{J_b} \le \chi_{j_b}$ *for* $1 \le b \le s$.

We write $\dim \mathcal{A} := \dim \pi_{[d]}\mathcal{A}$ by convention.

Proof of Theorem 3.2 assuming Theorem 3.1. For $n \in \mathbb{N}$ write $\Gamma_n = \vee_{i=0}^{n-1}\sigma^{-iN}\Gamma$. Note that $\mathcal{A} = \vee_{i=0}^{\infty}\sigma^{-i(nN)}\Gamma_n$ for all $n \in \mathbb{N}$. Applying Theorem 3.1 with $nN, \mathcal{C}_{nN}, \Gamma_n$ in place of $N, \mathcal{C}, \Gamma$, and taking $k = 0$, $l = s$, $J = J_s$, it follows that for β-a.e. y, the measure $\pi_J \Pi \beta_y^{\mathcal{A}}$ is exact dimensional with

$$\dim \pi_J \Pi \beta_y^{\mathcal{A}} = \sum_{b=1}^{s} \frac{\mathrm{H}^{\mathcal{C},\mathcal{A},n}_{J_{b-1}} - \mathrm{H}^{\mathcal{C},\mathcal{A},n}_{J_b}}{\chi_{j_b}} \qquad \text{for all } n \in \mathbb{N},$$

where for $I \subset [d]$,

$$\mathrm{H}^{\mathcal{C},\mathcal{A},n}_{I} = \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right).$$

For $1 \le b \le s$, applying Theorem 3.1 with $k = b-1, l = b$, we have

$$\mathrm{H}^{\mathcal{C},\mathcal{A},n}_{J_{b-1}} - \mathrm{H}^{\mathcal{C},\mathcal{A},n}_{J_b} \le \chi_{j_b},$$

since $\pi_{J_b}\Pi\beta_{y,x}^{\mathcal{A},\xi_{J_{b-1}}}$ is supported on $\Pi(x) + \pi_{j_b}\mathbb{R}^d$ for β-a.e. y and $\beta_y^{\mathcal{A}}$-a.e. x.

For $m, n \in \mathbb{N}$, it follows from $\mathcal{C}_{(m+n)N} \prec \mathcal{C}_{mN} \vee T^{-m}\mathcal{C}_{nN}$, $\widehat{\mathcal{A}} = \left(\vee_{i=0}^{m-1} T^{-i}\widehat{\Gamma}\right) \vee T^{-m}\widehat{\mathcal{A}}$, Lemmas 2.1, 2.2 and 3.5(i) that,

$$\begin{aligned}
&H\left(\beta, \mathcal{C}_{(m+n)N} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) \\
&\quad\le H\left(\beta, \mathcal{C}_{mN} \vee T^{-m}\mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) \\
&\quad= H\left(\beta, \mathcal{C}_{mN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) + H\left(\beta, T^{-m}\mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I \vee \widehat{\mathcal{C}_{mN}}\right) \\
&\quad= H\left(\beta, \mathcal{C}_{mN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) + H\left(\beta, T^{-m}\mathcal{C}_{nN} \mid \left(\vee_{i=0}^{m-1} T^{-i}\widehat{\Gamma}\right) \vee T^{-m}\widehat{\mathcal{A}} \vee T^{-m}\widehat{\xi}_I \vee \widehat{\mathcal{C}_{mN}}\right) \\
&\quad\le H\left(\beta, \mathcal{C}_{mN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) + H\left(\beta, T^{-m}\mathcal{C}_{nN} \mid T^{-m}\left(\widehat{\mathcal{A}} \vee \widehat{\xi}_I\right)\right) \\
&\quad= H\left(\beta, \mathcal{C}_{mN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right) + H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}} \vee \widehat{\xi}_I\right).
\end{aligned} \tag{3.6}$$

This shows the subadditivity and justifies the limit in (3.5). The proof is finished by letting $n \to \infty$ in the above equations. □

The rest of this section is devoted to the proof of Theorem 3.1. For the remainder of this section, we fix $N, \mathcal{C}, \Gamma, T, \mathcal{A}$ as in Theorem 3.1. Without loss of generality, we assume $J = [d]$, since the general case can be reduced to this one by considering the IFS Φ_J as defined in (2.1).

3.1. **The Peyrière measure.** We begin by introducing a useful measure on $\Omega \times \Lambda^{\mathbb{N}}$. Recall the definitions of $\Omega, \mathbf{P}, \beta^{\omega}, \mu^{\omega}$ from Section 2.4. Define a Borel probability measure $\mathbf{Q}$ on $\Omega \times \Lambda^{\mathbb{N}}$ by

$$\int_{\Omega \times \Lambda^{\mathbb{N}}} f(\omega, x)\, \mathrm{d}\mathbf{Q} = \int_{\Omega} \int_{\Lambda^{\mathbb{N}}} f(\omega, x)\, \mathrm{d}\beta^{\omega}(x) \mathrm{d}\mathbf{P}(\omega), \tag{3.7}$$

for every bounded Borel measurable function f on $\Omega \times \Lambda^{\mathbb{N}}$. Under this definition, the phrase "for $\mathbf{Q}$-a.e. (ω, x)" is equivalent to "for $\mathbf{P}$-a.e. ω and β^ω-a.e. x". The measure $\mathbf{Q}$ serves a role analogous to the Peyrière measure used in [15]. Next, define a transformation on $\Omega \times \Lambda^{\mathbb{N}}$ by

$$T(\omega, x) := (T\omega, Tx),$$

for $(\omega, x) \in \Omega \times \Lambda^{\mathbb{N}}$.

Lemma 3.3. *The system $(\Omega \times \Lambda^{\mathbb{N}}, \mathbf{Q}, T)$ is measure-preserving and mixing.*

Proof. For $A \in \mathcal{B}(\Omega \times \Lambda^{\mathbb{N}})$,

$$\begin{aligned}
\mathbf{Q}(T^{-1}A) &= \int_\Omega \int_{\Lambda^{\mathbb{N}}} \mathbf{1}_A(T\omega, Tx)\, \mathrm{d}\beta^\omega(x) \mathrm{d}\mathbf{P}(\omega) && \text{(by (3.7))}\\
&= \int_\Omega \int_{\Lambda^{\mathbb{N}}} \mathbf{1}_A(T\omega, x)\, \mathrm{d}\beta^{T\omega}(x) \mathrm{d}\mathbf{P}(\omega) && \text{(by (2.10))}\\
&= \int_\Omega \int_{\Lambda^{\mathbb{N}}} \mathbf{1}_A(\omega, x)\, \mathrm{d}\beta^\omega(x) \mathrm{d}\mathbf{P}(\omega) && \text{(by } T\mathbf{P} = \mathbf{P})\\
&= \mathbf{Q}(A). && \text{(by (3.7))}
\end{aligned}$$

Thus $\mathbf{Q}$ is T-invariant.

For $U \times I \in \Gamma^{m_1} \times (\Lambda^N)^{m_1}$, $V \times J \in \Gamma^{m_2} \times (\Lambda^N)^{m_2}, m_1, m_2 \geq 1$ and $n \geq 2Nm_1$, we have

$$\begin{aligned}
&\mathbf{Q}\left(([U] \times [I]) \cap T^{-n}([V] \times [J])\right)\\
&= \mathbf{Q}\left(([U] \cap T^{-n}[V]) \times ([I] \cap T^{-n}[J])\right)\\
&= \int_{[U] \cap T^{-n}[V]} \beta^\omega([I] \cap T^{-n}[J])\, \mathrm{d}\mathbf{P}(\omega) && \text{(by (3.7))}\\
&= \int_{[U] \cap T^{-n}[V]} \beta^\omega([I]) \beta^{T^n\omega}([J])\, \mathrm{d}\mathbf{P}(\omega) && \text{(by (2.7) and (2.10))}\\
&= \int_{[U]} \beta^\omega([I])\, \mathrm{d}\mathbf{P}(\omega) \int_{[V]} \beta^\omega([J])\, \mathrm{d}\mathbf{P}(\omega) && \text{(by (2.6))}\\
&= \mathbf{Q}\left([U] \times [I]\right) \mathbf{Q}\left([V] \times [J]\right). && \text{(by (3.7))}
\end{aligned}$$

This implies that T is mixing with respect to $\mathbf{Q}$. □

Below is a direct consequence of Birkhoff's ergodic theorem applied to $(\Omega \times \Lambda^{\mathbb{N}}, \mathbf{Q}, T)$.

Lemma 3.4. *For $\mathbf{Q}$-a.e. (ω, x) and $1 \leq j \leq d$, $\lim_{n\to\infty} -(1/n) \log \lambda_j^{x|nN} = N\chi_j$.*

3.2. **Some measurable partitions.** In this subsection we explore the properties of $\xi_{[j]}, \mathcal{A}$ and their associated conditional measures.

For $0 \leq j \leq d$, we denote $\xi_j = \xi_{[j]}$, $\Pi_j = \pi_{[j]}\Pi$, and for $x \in \Lambda^{\mathbb{N}}$, $r > 0$, define

$$B^{\Pi_j}(x, r) = \left\{ y \in \Lambda^{\mathbb{N}} \colon |\Pi_j(x) - \Pi_j(y)| \leq r \right\} = \Pi_j^{-1} B\left(\Pi_j x, r\right).$$

For $n \in \mathbb{N}$, let $\mathcal{C}_0^{n-1} := \vee_{i=0}^{n-1} T^{-i}\mathcal{C}$.

We begin with a lemma connecting ξ_j, $\mathcal{C}$ and $B^{\Pi_j}(x, r)$.

Lemma 3.5. *For $\mathbf{Q}$-a.e. (ω, x) and $1 \leq i \leq j \leq d$, the following holds.*

(i) $\xi_j(x)\cap\mathcal{C}(x)=T^{-1}\xi_j(Tx)\cap\mathcal{C}(x)$, *and so* $\xi_j\vee\mathcal{C}=T^{-1}\xi_j\vee\mathcal{C}$.

(ii) $\xi_{j-1}(x)\cap B^{\Pi_j}\left(x,\lambda_j^{x|nN}\right)\cap\mathcal{C}(x)=T^{-1}\left(\xi_{j-1}(Tx)\cap B^{\Pi_j}\left(Tx,\lambda_j^{Tx|(n-1)N}\right)\right)\cap\mathcal{C}(x)$.

(iii) *For* $\varepsilon\in(0,1)$ *and* $n\in\mathbb{N}$ *with* $\varepsilon^{-1}\ll n$, $\xi_{i-1}(x)\cap\mathcal{C}_0^{n-1}(x)\subset B^{\Pi_j}(x,\exp(-n(N\chi_i-\varepsilon)))$.

Proof. By (1.15),

$$\varphi_{x|nN}\left(\Pi(T^n x)\right)=\Pi(x)\quad\text{for } x\in\Lambda^{\mathbb{N}},n\in\mathbb{N}. \tag{3.8}$$

For $x\in\Lambda^{\mathbb{N}}$, $n\in\mathbb{N}$, $a,b\in\mathbb{R}^d$ and $J\subset[d]$, since $A_{\varphi_{x|n}}$ is a diagonal matrix, we have

$$\pi_J(\varphi_{x|n}(a)-\varphi_{x|n}(b))=\varphi_{x|n}(\pi_J a)-\varphi_{x|n}(\pi_J b). \tag{3.9}$$

Then for $y\in\mathcal{C}(x)$, we have $\varphi_{x|N}=\varphi_{y|N}$, and so

$$\begin{aligned}
y\in\xi_j(x)&\iff \pi_{[j]}\Pi(x)=\pi_{[j]}\Pi(y) && \text{(by (3.1))}\\
&\iff \pi_{[j]}\varphi_{x|N}\left(\Pi(Tx)\right)=\pi_{[j]}\varphi_{y|N}\left(\Pi(Ty)\right) && \text{(by (3.8))}\\
&\iff \pi_{[j]}\varphi_{x|N}\left(\Pi(Tx)\right)=\pi_{[j]}\varphi_{x|N}\left(\Pi(Ty)\right) && \text{(by } \varphi_{x|N}=\varphi_{y|N})\\
&\iff \varphi_{x|N}\left(\pi_{[j]}\Pi(Tx)\right)=\varphi_{x|N}\left(\pi_{[j]}\Pi(Ty)\right) && \text{(by (3.9))}\\
&\iff \pi_{[j]}\Pi(Tx)=\pi_{[j]}\Pi(Ty) && \text{(by } \varphi_{x|N} \text{ being invertible)}\\
&\iff y\in T^{-1}\xi_j(Tx). && \text{(by (3.1))}
\end{aligned}$$

This proves (i).

For $y\in\mathcal{C}(x)$, we have $\varphi_{x|N}=\varphi_{y|N}$, and so

$$\begin{aligned}
&y\in\xi_{j-1}(x)\cap B^{\Pi_j}\left(x,\lambda_j^{x|nN}\right)\\
\iff& \left|\pi_{[j]}\Pi(x)-\pi_{[j]}\Pi(y)\right|\le\lambda_j^{x|nN},\ \pi_{[j-1]}\Pi(x)=\pi_{[j-1]}\Pi(y)\\
\iff& |\pi_j\Pi(x)-\pi_j\Pi(y)|\le\lambda_j^{x|nN},\ \pi_{[j-1]}\Pi(x)=\pi_{[j-1]}\Pi(y)\quad \text{(by } \pi_{[j]}=\pi_{[j-1]}+\pi_j)\\
\iff& \left|\pi_j\varphi_{x|N}(\Pi(Tx))-\pi_j\varphi_{x|N}(\Pi(Ty))\right|\le\lambda_j^{x|nN}, \qquad \text{(by (3.8) and } \varphi_{x|N}=\varphi_{y|N})\\
&\qquad \pi_{[j-1]}\Pi(Tx)=\pi_{[j-1]}\Pi(Ty) \qquad \text{(by (i))}\\
\iff& \lambda_j^{x|N}\left|\pi_j\Pi(Tx)-\pi_j\Pi(Ty)\right|\le\lambda_j^{x|nN},\ \pi_{[j-1]}\Pi(Tx)=\pi_{[j-1]}\Pi(Ty)\quad\text{(by (3.9))}\\
\iff& |\pi_j\Pi(Tx)-\pi_j\Pi(Ty)|\le\lambda_j^{Tx|(n-1)N},\ \pi_{[j-1]}\Pi(Tx)=\pi_{[j-1]}\Pi(Ty)\\
\iff& \left|\pi_{[j]}\Pi(Tx)-\pi_{[j]}\Pi(Ty)\right|\le\lambda_j^{Tx|(n-1)N},\ \pi_{[j-1]}\Pi(Tx)=\pi_{[j-1]}\Pi(Ty)\\
\iff& y\in T^{-1}B^{\Pi_j}\left(Tx,\lambda_j^{Tx|(n-1)N}\right)\cap T^{-1}\xi_{j-1}(Tx).
\end{aligned}$$

This gives (ii).

Finally, we prove (iii). By Lemma 3.4 and $\chi_\ell\ge\chi_i$, we have for $\mathbf{Q}$-a.e. (ω,x) and $i\le\ell\le j$,

$$\lambda_\ell^{x|nN}\le\exp\left(-n(N\chi_\ell-\varepsilon/4)\right)\le\exp\left(-n(N\chi_i-\varepsilon/2)\right). \tag{3.10}$$

Let $y\in\mathcal{C}_0^{n-1}\cap\xi_{i-1}(x)$. Then $\varphi_{y|nN}=\varphi_{x|nN}$ and $\pi_{[i-1]}\Pi(x)=\pi_{[i-1]}\Pi(y)$. Hence

$$\begin{aligned}
&\left|\pi_{[j]}\Pi(x)-\pi_{[j]}\Pi(y)\right|\\
&\qquad=\left|\sum_{\ell=i}^{j}\pi_\ell\Pi(x)-\pi_\ell\Pi(y)\right| \qquad\qquad \text{(by } \pi_{[i-1]}\Pi(x)=\pi_{[i-1]}\Pi(y))
\end{aligned}$$

$$= \left| \sum_{\ell=i}^{j} \pi_\ell \left(\varphi_{x|nN} \Pi(T^n x) - \varphi_{x|nN} \Pi(T^n y) \right) \right| \qquad \text{(by (3.8) and } \varphi_{y|nN} = \varphi_{x|nN})$$

$$\leq \sum_{\ell=i}^{j} \lambda_\ell^{x|nN} \qquad \text{(by } \operatorname{diam}(K_\Phi) \leq 1)$$

$$\leq \exp\left(-n(N\chi_i - \varepsilon)\right). \qquad \text{(by (3.10))}$$

This shows that $y \in B^{\Pi_j}(x, \exp(-n(N\chi_i - \varepsilon)))$. □

Next, we establish the relation between the conditional measures $\beta_x^{\omega,\xi_j} := (\beta^\omega)_x^{\xi_j}$ and $\beta_{Tx}^{T\omega,\xi_j}$.

Lemma 3.6. *For* $\mathbf{Q}$*-a.e.* (ω, x)*,* $1 \leq j \leq d$ *and* $A \subset \mathcal{B}(\Lambda^{\mathbb{N}})$*,*

$$\beta_{Tx}^{T\omega,\xi_j}(A) = \frac{\beta_x^{\omega,\xi_j}(T^{-1}A \cap \mathcal{C}(x))}{\beta_x^{\omega,\xi_j}(\mathcal{C}(x))}.$$

Proof. First we show that

$$\begin{aligned}
\beta_x^{\omega,T^{-1}\xi_j \vee \mathcal{C}}(T^{-1}A) &= \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A} \mid T^{-1}\widehat{\xi_j} \vee \mathcal{C}\right)(x) && \text{(by Theorem 2.3)} \\
&= \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A} \mid T^{-1}\widehat{\xi_j}\right)(x) && \text{(by Lemma 2.1(vi))} \\
&= \mathbf{E}\left(\beta^{T\omega}, \mathbf{1}_{A} \mid \widehat{\xi_j}\right)(Tx) && \text{(by Lemma 2.1(i) and (2.10))} \\
&= \beta_{Tx}^{T\omega,\xi_j}(A). && \text{(by Theorem 2.3)}
\end{aligned} \tag{3.11}$$

By Theorem 2.3, for β-a.e. x we define

$$\nu_x(T^{-1}A) = \frac{\beta_x^{\omega,\xi_j}(T^{-1}A \cap \mathcal{C}(x))}{\beta_x^{\omega,\xi_j}(\mathcal{C}(x))} = \sum_{B \in \mathcal{C}} \mathbf{1}_B(x) \cdot h_B(x),$$

where $h_B := \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A \cap B} \mid \widehat{\xi_j}\right) / \mathbf{E}\left(\beta^\omega, \mathbf{1}_B \mid \widehat{\xi_j}\right)$. Since h_B is $\widehat{\xi_j}$ measurable, the function $x \mapsto \nu_x(T^{-1}A)$ is $\widehat{\xi_j} \vee \widehat{\mathcal{C}}$-measurable. Moreover,

$$\begin{aligned}
\int \nu_x(T^{-1}A)\, \mathrm{d}\beta^\omega &= \sum_{B \in \mathcal{C}} \int \mathbf{1}_B h_B \, \mathrm{d}\beta^\omega \\
&= \sum_{B \in \mathcal{C}} \int \mathbf{E}\left(\beta^\omega, \mathbf{1}_B h_B \mid \widehat{\xi_j}\right) \mathrm{d}\beta^\omega \\
&= \sum_{B \in \mathcal{C}} \int \mathbf{E}\left(\beta^\omega, \mathbf{1}_B \mid \widehat{\xi_j}\right) h_B \, \mathrm{d}\beta^\omega && \text{(by } h_B \text{ being } \widehat{\xi_j}\text{-measurable)} \\
&= \sum_{B \in \mathcal{C}} \int \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A \cap B} \mid \widehat{\xi_j}\right) \mathrm{d}\beta^\omega && \text{(by the definition of } h_B) \\
&= \sum_{B \in \mathcal{C}} \beta^\omega(T^{-1}A \cap B) = \beta^\omega(T^{-1}A).
\end{aligned} \tag{3.12}$$

Hence, the uniqueness of conditional expectation implies that

$$\begin{aligned}
\nu_x(T^{-1}A) &= \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A} \mid \widehat{\xi_j} \vee \widehat{\mathcal{C}}\right) \\
&= \mathbf{E}\left(\beta^\omega, \mathbf{1}_{T^{-1}A} \mid T^{-1}\widehat{\xi_j} \vee \widehat{\mathcal{C}}\right) && \text{(by Lemma 3.5(i))}
\end{aligned}$$

$$= \beta_x^{\omega, T^{-1}\xi_j \vee \mathcal{C}}(T^{-1}A). \qquad \text{(by Theorem 2.3)}$$

This together with (3.11) finishes the proof. $\square$

Then we compute some useful integrals related to the conditional information and entropy.

Lemma 3.7. *Let $\mathcal{E}$ be a finite partition of $\Lambda^{\mathbb{N}}$, and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{B}(\Lambda^{\mathbb{N}})$. Then*

$$\int_{\Omega\times\Lambda^{\mathbb{N}}} \mathbf{I}(\beta^\omega, \mathcal{E} \mid \mathcal{F})(x)\, \mathrm{d}\mathbf{Q}(\omega, x) = H\left(\beta, \mathcal{E} \mid \widehat{\mathcal{A}} \vee \mathcal{F}\right), \tag{3.13}$$

and

$$\int_{\Omega} H(\beta^\omega, \mathcal{E} \mid \mathcal{F})\, \mathrm{d}\mathbf{P}(\omega) = H\left(\beta, \mathcal{E} \mid \widehat{\mathcal{A}} \vee \mathcal{F}\right). \tag{3.14}$$

Proof. Since $(\Lambda^{\mathbb{N}}, \mathcal{B}(\Lambda^{\mathbb{N}}), \beta)$ is a separable probability space, there exists a sequence of countable partitions $(\mathcal{F}_n)_{n=1}^\infty$ of $\Lambda^{\mathbb{N}}$ so that $\widehat{\mathcal{F}_n} \uparrow \mathcal{F}$. Note that for any sub-$\sigma$-algebra $\mathcal{G}$ of $\mathcal{B}(\Lambda^{\mathbb{N}})$,

$$\begin{aligned}
&\int_{\Omega\times\Lambda^{\mathbb{N}}} \mathbf{I}(\beta^\omega, \mathcal{E} \mid \mathcal{G})(x)\, \mathrm{d}\mathbf{Q}(\omega, x) \\
&= \int_{\Omega} H(\beta^\omega, \mathcal{E} \mid \mathcal{G})\, \mathrm{d}\mathbf{P}(\omega) && \text{(3.15)} \quad \text{(by (3.7))} \\
&= \int_{\Lambda^{\mathbb{N}}} H\left(\beta_y^{\mathcal{A}}, \mathcal{E} \mid \mathcal{G}\right)\, \mathrm{d}\beta(y) && \text{(3.16)} \quad \text{(by (2.8))} \\
&= \int_{\Lambda^{\mathbb{N}}} \int_{\Lambda^{\mathbb{N}}} \mathbf{I}\left(\beta_y^{\mathcal{A}}, \mathcal{E} \mid \mathcal{G}\right)(x)\, \mathrm{d}\beta_y^{\mathcal{A}}(x) \mathrm{d}\beta(y) && \text{(by (2.4))} \\
&= \int_{\Lambda^{\mathbb{N}}} \int_{\Lambda^{\mathbb{N}}} \mathbf{I}\left(\beta_x^{\mathcal{A}}, \mathcal{E} \mid \mathcal{G}\right)(x)\, \mathrm{d}\beta_y^{\mathcal{A}}(x) \mathrm{d}\beta(y) && \text{(by } \beta_x^{\mathcal{A}} = \beta_y^{\mathcal{A}} \text{ if } x \in \mathcal{A}(y)) \\
&= \int_{\Lambda^{\mathbb{N}}} \mathbf{I}\left(\beta_x^{\mathcal{A}}, \mathcal{E} \mid \mathcal{G}\right)(x)\, \mathrm{d}\beta(x). && \text{(3.17)} \quad \text{(by (2.8))}
\end{aligned}$$

Since (3.14) follows from (3.13) and (3.15), it suffices to prove (3.13).

For each $E \in \mathcal{E}$, $n \in \mathbb{N}$, β-a.e. x, by Theorem 2.3 we have

$$\mathbf{E}\left(\beta_x^{\mathcal{A}}, \mathbf{1}_E \mid \widehat{\mathcal{F}_n}\right)(x) = \frac{\beta_x^{\mathcal{A}}(E \cap \mathcal{F}_n(x))}{\beta_x^{\mathcal{A}}(\mathcal{F}_n(x))} = \sum_{F \in \mathcal{F}_n} \mathbf{1}_F(x) h_F(x),$$

where $h_F(x) = \mathbf{E}\left(\beta, \mathbf{1}_{E\cap F} \mid \widehat{\mathcal{A}}\right) / \mathbf{E}\left(\beta, \mathbf{1}_F \mid \widehat{\mathcal{A}}\right)$. Then $x \mapsto \mathbf{E}\left(\beta_x^{\mathcal{A}}, \mathbf{1}_E \mid \widehat{\mathcal{F}_n}\right)(x)$ is $\widehat{\mathcal{A}} \vee \widehat{\mathcal{F}_n}$ measurable. This together with the computation in (3.12) shows that

$$\mathbf{E}\left(\beta_x^{\mathcal{A}}, \mathbf{1}_E \mid \widehat{\mathcal{F}_n}\right)(x) = \mathbf{E}\left(\beta, \mathbf{1}_E \mid \widehat{\mathcal{A}} \vee \widehat{\mathcal{F}_n}\right)(x). \tag{3.18}$$

Hence

$$\begin{aligned}
&\int_{\Omega\times\Lambda^{\mathbb{N}}} \mathbf{I}(\beta^\omega, \mathcal{E} \mid \mathcal{F})(x)\, \mathrm{d}\mathbf{Q}(\omega, x) \\
&= \int_{\Lambda^{\mathbb{N}}} H\left(\beta_y^{\mathcal{A}}, \mathcal{E} \mid \mathcal{F}\right)\, \mathrm{d}\beta(y) && \text{(by (3.16))} \\
&= \int_{\Lambda^{\mathbb{N}}} \lim_{n\to\infty} H\left(\beta_y^{\mathcal{A}}, \mathcal{E} \mid \widehat{\mathcal{F}_n}\right)\, \mathrm{d}\beta(y) && \text{(by Lemma 2.1(viii) and } \#\mathcal{E} < \infty) \\
&= \lim_{n\to\infty} \int_{\Lambda^{\mathbb{N}}} H\left(\beta_y^{\mathcal{A}}, \mathcal{E} \mid \widehat{\mathcal{F}_n}\right)\, \mathrm{d}\beta(y) && \text{(by } \#\mathcal{E} < \infty)
\end{aligned}$$

$$= \lim_{n\to\infty} \int_{\Lambda^{\mathbb{N}}} \mathbf{I}\left(\beta_x^{\mathcal{A}}, \mathcal{E} \mid \widehat{\mathcal{F}_n}\right)(x)\, \mathrm{d}\beta(x) \qquad \text{(by (3.17))}$$
$$= \lim_{n\to\infty} H\left(\beta, \mathcal{E} \mid \widehat{\mathcal{A}} \vee \widehat{\mathcal{F}_n}\right) \qquad \text{(by (3.18))}$$
$$= H\left(\beta, \mathcal{E} \mid \widehat{\mathcal{A}} \vee \mathcal{F}\right), \qquad \text{(by Lemma 2.1(viii) and } \#\mathcal{E} < \infty)$$

which finishes the proof. □

We finish this subsection with the a version of Shannon-McMillan-Breiman theorem.

Lemma 3.8. *For* $\mathbf{Q}$*-a.e.* (ω, x) *and* $0 \le j \le d$, $\lim_{n\to\infty} -(1/n)\log \beta_x^{\omega,\xi_j}(\mathcal{C}_0^{n-1}(x)) = N\mathrm{H}_{[j]}^{\mathcal{A}}$.

Proof. For $n \in \mathbb{N}$, we have

$$\mathbf{I}\left(\beta^\omega, \vee_{i=0}^{n-1} T^{-i}\mathcal{C} \mid \widehat{\xi_j}\right)(x)$$
$$= \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x) + \mathbf{I}\left(\beta^\omega, \vee_{i=1}^{n-1} T^{-i}\mathcal{C} \mid \widehat{\xi_j} \vee \widehat{\mathcal{C}}\right)(x) \qquad \text{(by Lemma 2.1(iv))}$$
$$= \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x) + \mathbf{I}\left(\beta^\omega, \vee_{i=1}^{n-1} T^{-i}\mathcal{C} \mid T^{-1}\widehat{\xi_j} \vee \widehat{\mathcal{C}}\right)(x) \qquad \text{(by Lemma 3.5(i))}$$
$$= \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x) + \mathbf{I}\left(\beta^\omega, \vee_{i=1}^{n-1} T^{-i}\mathcal{C} \mid T^{-1}\widehat{\xi_j}\right)(x) \qquad \text{(by Lemma 2.1(vii))}$$
$$= \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x) + \mathbf{I}\left(\beta^{T\omega}, \vee_{i=0}^{n-2} T^{-i}\mathcal{C} \mid \widehat{\xi_j}\right)(Tx). \qquad \text{(by Lemma 2.1(ii) and (2.10))}$$

Then an induction shows that

$$\mathbf{I}\left(\beta^\omega, \vee_{i=0}^{n-1} T^{-i}\mathcal{C} \mid \widehat{\xi_j}\right)(x) = \sum_{k=0}^{n-1} \mathbf{I}\left(\beta^{T^k\omega}, \mathcal{C} \mid \widehat{\xi_j}\right)(T^k x). \tag{3.19}$$

On the other hand, it follows from Theorem 2.3 and (2.3) that for $\mathbf{Q}$-a.e. (ω, x),

$$-\log \beta_x^{\omega,\xi_j}(\mathcal{C}_0^{n-1}(x)) = \mathbf{I}\left(\beta^\omega, \vee_{i=0}^{n-1} T^{-i}\mathcal{C} \mid \widehat{\xi_j}\right)(x). \tag{3.20}$$

By (3.19), (3.20) and (3.13), applying Birkhoff's ergodic theorem finishes the proof. □

3.3. **Transverse dimensions.** The aim of this subsection is to prove Proposition 3.9, which intuitively provides the local dimension of μ^ω along each coordinate.

Proposition 3.9. *For* $\mathbf{Q}$*-a.e.* (ω, x) *and* $1 \le j \le d$,

$$\lim_{r\to 0} \frac{\log \beta_x^{\omega,\xi_{j-1}}(B^{\Pi_j}(x,r))}{\log r} = \frac{\mathrm{H}_{[j-1]}^{\mathcal{A}} - \mathrm{H}_{[j]}^{\mathcal{A}}}{\chi_j},$$

where $\mathrm{H}_I^{\mathcal{A}}$ *is defined in* (3.3).

The proof of Proposition 3.9 is inspired by [17, Proposition 5.1]. The key idea is to reformulate the measures of small balls in terms of certain variants of Birkhoff sums. The proof is then completed by applying Birkhoff's and the following Maker's ergodic theorems [38].

Lemma 3.10 (Maker [38]). *Let* T *be a measure-preserving transformation on a probability space* $(X, \mathcal{B}, \theta)$. *Let* $(g_n)_{n=1}^\infty$ *be a sequence of measurable functions converging* θ*-a.e. to* g. *Suppose* $\sup_n |g_n| \le f$ *for some* $f \in L^1(X, \mathcal{B}, \theta)$. *Then both* θ*-a.e. and in* L^1,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g_{n-k}(T^k x) = \mathbf{E}(\theta, g \mid \mathcal{I})(x),$$

where $\mathcal{I} = \{B \in \mathcal{B}, T^{-1}B = B\}$.

The following lemma is a preparation for applying Lemma 3.10.

Lemma 3.11. *For* $\mathbf{Q}$*-a.e.* (ω, x) *and* $1 \le j \le d$,

$$\lim_{r\to 0} -\log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\cap \mathcal{C}(x)\right)}{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\right)} = \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x). \tag{3.21}$$

Furthermore, set

$$g(\omega, x) = -\inf_{r>0} \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\cap \mathcal{C}(x)\right)}{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\right)}.$$

Then $g \ge 0$ *and* $g \in L^1(\Omega \times \Lambda^{\mathbb{N}}, \mathbf{Q})$.

Proof. Applying [17, Lemma 2.5(2)] with $\Lambda^{\mathbb{N}}, \pi_{[j]}\mathbb{R}^d, \pi_{[j]}, \beta^\omega, \mathcal{C}, \xi_{j-1}$ in place of $X, Y, \pi, m, \alpha, \eta$ gives

$$\lim_{r\to 0} -\log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\cap \mathcal{C}(x)\right)}{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\right)} = \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j} \vee \widehat{\xi_{j-1}}\right)(x).$$

This implies (3.21) since $\xi_{j-1} \prec \xi_j$. The last statement follows from the second part of [17, Lemma 2.5(2)] and $H(\beta^\omega, \mathcal{C}) \le N \log|\Lambda|$ for all $\omega \in \Omega$. □

We are now ready to prove Proposition 3.9.

Proof of Proposition 3.9. The proof is adapted from [17, Proposition 5.1]. For clarity and to account for the dependence on ω, we provide the details in full.

For $n \in \mathbb{N}$, define

$$H_n(\omega, x) = \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}\left(x, \lambda_j^{x|nN}\right)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}\left(Tx, \lambda_j^{Tx|(n-1)N}\right)\right)}. \tag{3.22}$$

Then by telescoping and $\operatorname{diam}(\operatorname{supp}\mu) \le 1$,

$$\sum_{k=0}^{n-1} H_{n-k}(T^k(\omega, x)) = \log \beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}\left(x, \lambda_j^{x|nN}\right)\right). \tag{3.23}$$

For $n \in \mathbb{N}$, define

$$G_n(\omega, x) = \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}\left(x, \lambda_j^{x|nN}\right)\cap \mathcal{C}(x)\right)}{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}\left(x, \lambda_j^{x|nN}\right)\right)}. \tag{3.24}$$

For $1 \le j \le d$, write

$$Q_j(\omega, x) = \mathbf{I}\left(\beta^\omega, \mathcal{C} \mid \widehat{\xi_j}\right)(x). \tag{3.25}$$

Then Lemma 3.11 implies that $\sup_n |G_n| \in L^1(\mathbf{Q})$ and for $\mathbf{Q}$-a.e. (ω, x),

$$\lim_{n\to\infty} G_n = -Q_j.$$

Thus for $\mathbf{Q}$-a.e. (ω, x), combining Lemma 3.10 and Lemma 3.7 shows that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} G_{n-k}(T^k(\omega,x)) = -\int Q_j \,\mathrm{d}\mathbf{Q} = -N\mathrm{H}^{\mathcal{A}}_{[j]}, \tag{3.26}$$

and by Birkhoff's ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} Q_{j-1}(T^k(\omega,x)) = N\mathrm{H}^{\mathcal{A}}_{[j-1]}. \tag{3.27}$$

Next, we show that for $n \in \mathbb{N}$,

$$H_n = -Q_{j-1} - G_n. \tag{3.28}$$

This is justified as follows,

$$\begin{aligned}
&H_n(\omega,x) + G_n(\omega,x) \\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,\lambda_j^{x|nN})\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by (3.22) and (3.24))}\\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(\xi_{j-1}(x)\cap B^{\Pi_j}(x,\lambda_j^{x|nN})\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by } \beta_x^{\omega,\xi_{j-1}}(\xi_{j-1}(x))=1)\\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(T^{-1}\left(\xi_{j-1}(Tx)\cap B^{\Pi_{[j]}}(Tx,\lambda_j^{Tx|(n-1)N})\right)\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by Lemma 3.5(ii))}\\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(T^{-1}B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\cap T^{-1}\xi_{j-1}(Tx)\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by rearranging)}\\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(T^{-1}B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\cap \xi_{j-1}(x)\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by Lemma 3.5(i))}\\
&= \log \frac{\beta_x^{\omega,\xi_{j-1}}\left(T^{-1}B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\cap \mathcal{C}(x)\right)}{\beta_{Tx}^{T\omega,\xi_{j-1}}\left(B^{\Pi_j}(Tx,\lambda_j^{Tx|(n-1)N})\right)} && \text{(by } \beta_x^{\omega,\xi_{j-1}}(\xi_{j-1}(x))=1)\\
&= \log \beta_x^{\omega,\xi_{j-1}}\left(\mathcal{C}(x)\right) && \text{(by Lemma 3.6)}\\
&= -\mathbf{I}\left(\beta^{\omega}, \mathcal{C} \mid \widehat{\xi_{j-1}}\right)(x) && \text{(by Theorem 2.3 and (2.3))}\\
&= -Q_{j-1}(\omega,x). && \text{(by (3.25))}
\end{aligned}$$

Finally, for $\mathbf{Q}$-a.e. (ω, x), we have

$$\begin{aligned}
&\lim_{r\to 0}\frac{\log \beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,r)\right)}{\log r}\\
&\quad= \lim_{n\to\infty}\frac{\log \beta_x^{\omega,\xi_{j-1}}\left(B^{\Pi_j}(x,\lambda_j^{x|nN})\right)}{\log \lambda_j^{x|nN}} && \text{(by Lemma 3.4)}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} H_{n-k}(T^k(\omega,x))}{\log \lambda_j^{x|nN}} \qquad \text{(by (3.23))}$$

$$= \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} Q_{j-1}(T^k(\omega,x)) + \sum_{k=0}^{n} G_{n-k}(T^k(\omega,x))}{-\log \lambda_j^{x|nN}} \qquad \text{(by (3.28))}$$

$$= \lim_{n\to\infty} \frac{H^{\mathcal{A}}_{[j-1]} - H^{\mathcal{A}}_{[j]}}{\chi_j}. \qquad \text{(by (3.26), (3.27) and Lemma 3.4)}$$

This finishes the proof. □

3.4. **Proof of Theorem 3.1.** In this subsection, we prove Theorem 3.1 by adapting the arguments in [17, Section 6], which is itself inspired by ideas from Ledrappier and Young [37].

For $1 \le i \le d$, denote

$$\vartheta_i := \frac{\mathrm{H}^{\mathcal{A}}_{[i-1]} - \mathrm{H}^{\mathcal{A}}_{[i]}}{\chi_i}. \tag{3.29}$$

Using Proposition 3.9, it follows that for $\mathbf{Q}$-a.e. (ω, x),

$$\vartheta_i = \lim_{r\to 0} \frac{\log \beta_x^{\omega,\xi_{i-1}}(B^{\Pi_i}(x,r))}{\log r}. \tag{3.30}$$

For $\mathbf{Q}$-a.e. (ω, x) and $0 \le i \le j \le d$, define

$$\overline{\gamma}^{\omega}_{i,j}(x) = \limsup_{r\to 0} \frac{\log \beta_x^{\omega,\xi_i}\left(B^{\Pi_j}(x,r)\right)}{\log r} \quad \text{and} \quad \underline{\gamma}^{\omega}_{i,j} = \liminf_{r\to 0} \frac{\log \beta_x^{\omega,\xi_i}\left(B^{\Pi_j}(x,r)\right)}{\log r}. \tag{3.31}$$

We claim that the following three statements hold for $\mathbf{Q}$-a.e. (ω, x):

(D1) $$\overline{\gamma}^{\omega}_{j,j}(x) = \underline{\gamma}^{\omega}_{j,j}(x) = 0$$

(D2) $$\chi_i \left(\overline{\gamma}^{\omega}_{i-1,j}(x) - \overline{\gamma}^{\omega}_{i,j}(x)\right) \le \mathrm{H}^{\mathcal{A}}_{[i-1]} - \mathrm{H}^{\mathcal{A}}_{[i]} \quad \text{for } 1 \le i \le j.$$

(D3) $$\underline{\gamma}^{\omega}_{i,j}(x) + \vartheta_i \le \underline{\gamma}^{\omega}_{i-1,j}(x) \quad \text{for } 1 \le i \le j.$$

Proof of Theorem 3.1 assuming (D1)–(D3). Combining (3.30), (D2) and (D3) shows that if $\overline{\gamma}^{\omega}_{i,j}(x) = \underline{\gamma}^{\omega}_{i,j}(x) = \gamma^{\omega}_{i,j}(x)$ for some $\gamma^{\omega}_{i,j}(x) \in \mathbb{R}$, then

$$\underline{\gamma}^{\omega}_{i-1,j}(x) \le \overline{\gamma}^{\omega}_{i-1,j}(x) \le \overline{\gamma}^{\omega}_{i,j}(x) + \theta_i = \underline{\gamma}^{\omega}_{i,j}(x) + \theta_i \le \underline{\gamma}^{\omega}_{i-1,j}(x). \tag{3.32}$$

Thus $\underline{\gamma}^{\omega}_{i-1,j}(x) = \overline{\gamma}^{\omega}_{i-1,j}(x) = \gamma^{\omega}_{i-1,j}(x)$ for some $\gamma^{\omega}_{i-1,j}(x) \in \mathbb{R}$, and so

$$\gamma^{\omega}_{i-1,j}(x) = \gamma^{\omega}_{i,j}(x) + \vartheta_i. \tag{3.33}$$

By (D1), an induction from $i = j$ shows that (3.32) and (3.33) hold for all $1 \le i \le j$. Hence

$$\gamma^{\omega}_{i,j}(x) = \sum_{\ell=i+1}^{j} \vartheta_\ell = \sum_{\ell=i+1}^{j} \frac{\mathrm{H}^{\mathcal{A}}_{[\ell-1]} - \mathrm{H}^{\mathcal{A}}_{[\ell]}}{\chi_\ell} \quad \text{for } 0 \le i \le j. \tag{3.34}$$

Note that for $\mathbf{Q}$-a.e. (ω, x) and $r > 0$,

$$\beta_x^{\omega,\xi_i}\left(B^{\Pi_j}(x,r)\right) = (\pi_{[j]}\Pi\beta_x^{\omega,\xi_i})\left(B\left(\pi_{[j]}\Pi(x), r\right)\right).$$

This together with (3.31) and (3.34) shows that for $\mathbf{Q}$-a.e. (ω, x) and $0 \le i \le j$, the measure $\pi_{[j]}\Pi\beta_x^{\omega,\xi_i}$ is exact dimensional with

$$\dim \pi_{[j]}\Pi\beta_x^{\omega,\xi_i} = \sum_{\ell=i+1}^{j} \frac{\mathrm{H}^{\mathcal{A}}_{[\ell-1]} - \mathrm{H}^{\mathcal{A}}_{[\ell]}}{\chi_\ell}. \tag{3.35}$$

This proves Theorem 3.1 when $J = [d]$. For general $J \subset [d]$, the proof is finished by considering Φ_J instead. □

It remains to prove (D1)–(D3).

Proof of (D1). Since $\xi_j(x) = \Pi_j^{-1}(\Pi_j(x)) \subset B^{\Pi_j}(x, r)$ for every $x \in \Lambda^{\mathbb{N}}$ and $r > 0$, we have

$$1 \ge \beta_x^{\omega,\xi_j}\left(B^{\Pi_j}(x, r)\right) \ge \beta_x^{\omega,\xi_j}(\xi_j(x)) = 1.$$

Thus $\overline{\gamma}^{\omega}_{j,j}(x) = \underline{\gamma}^{\omega}_{j,j}(x) = 0$ for $\mathbf{Q}$-a.e. (ω, x). □

The proof of (D2) and (D3) relies on the next lemma showing that a set with positive measure has positive density with respect to conditional measures almost surely.

Lemma 3.12. *Let $\omega \in \Omega$ and $A \in \mathcal{B}(\Lambda^{\mathbb{N}})$ be with $\beta^\omega(A) > 0$. Then for $0 \le i \le j \le d$ and β^ω-a.e. $x \in A$,*

$$\lim_{r \to 0} \frac{\beta_x^{\omega,\xi_i}(A \cap B^{\Pi_j}(x, r))}{\beta_x^{\omega,\xi_i}(B^{\Pi_j}(x, r))} > 0.$$

Proof. Applying [17, Lemma 2.5(1)] with $\Lambda^{\mathbb{N}}, \pi_{[j]}\mathbb{R}^d, \pi_{[j]}, \beta^\omega, \mathcal{C}, \xi_i$ in place of $X, Y, \pi, m, \alpha, \eta$ shows that for β^ω-a.e. x,

$$\lim_{r \to 0} \frac{\beta_x^{\omega,\xi_i}(A \cap B^{\Pi_j}(x, r))}{\beta_x^{\omega,\xi_i}(B^{\Pi_j}(x, r))} = \mathbf{E}\left(\beta^\omega, \mathbf{1}_A \mid \widehat{\xi}_i \vee \widehat{\xi}_j\right)(x).$$

The proof is completed by an almost trivial property of conditional expectation that, for a probability space $(X, \mathcal{B}, \theta)$ and a sub-σ-algebra $\mathcal{F}$ of $\mathcal{B}$, letting $A \in \mathcal{B}$ be with $\theta(A) > 0$, we have

$$\mathbf{E}(\theta, \mathbf{1}_A \mid \mathcal{F})(x) > 0 \quad \text{for } \theta\text{-a.e. } x \in A.$$

(See e.g. [19, Lemma 3.10] for a proof.) □

Now we are ready to prove (D2) and (D3).

Proof of (D2). For $0 \le i \le j$, write $h_i := \mathrm{H}^{\mathcal{A}}_{[i]}$ for short. Suppose on the contrary that (D2) is not true. There exists $1 \le i \le j$ and $U \subset \Omega \times \Lambda^{\mathbb{N}}$ with $\mathbf{Q}(U) > 0$ such that for $(\omega, x) \in U$,

$$\frac{h_{i-1} - h_i}{\chi_i} < \overline{\gamma}^{\omega}_{i-1,j}(x) - \overline{\gamma}^{\omega}_{i,j}(x). \tag{3.36}$$

It follows from (3.36) and (3.31) that U is a subset of the following set,

$$\bigcup_{\alpha \in \mathbb{Q} \cap (0,\infty)} \bigcup_{\overline{\gamma}_{i-1}, \overline{\gamma}_i \in \mathbb{Q}} \bigcap_{\varepsilon > 0} \Big\{ (\omega, x) \colon \frac{h_{i-1} - h_i}{\chi_i} < \overline{\gamma}_{i-1} - \overline{\gamma}_i - \alpha,$$
$$\overline{\gamma}^{\omega}_{i-1,j}(x) > \overline{\gamma}_{i-1} - \varepsilon/2,\ \overline{\gamma}^{\omega}_{i,j}(x) < \overline{\gamma}_i + \varepsilon/2 \Big\}.$$

Then there exist $\alpha > 0$ and real numbers $\overline{\gamma}_{i-1}, \overline{\gamma}_i$ such that

$$\frac{h_{i-1} - h_i}{\chi_i} < \overline{\gamma}_{i-1} - \overline{\gamma}_i - \alpha, \tag{3.37}$$

and for $\varepsilon > 0$ there exists $U_\varepsilon \subset U$ with $\mathbf{Q}(U_\varepsilon) > 0$ so that for $x \in U_\varepsilon$,

$$\overline{\gamma}^{\omega}_{i-1,j}(x) > \overline{\gamma}_{i-1} - \varepsilon/2, \qquad \overline{\gamma}^{\omega}_{i,j}(x) < \overline{\gamma}_i + \varepsilon/2. \tag{3.38}$$

Fix $\varepsilon \in (0, \chi_i/3)$. There exists $n_0 \colon U_\varepsilon \to \mathbb{N}$ such that for $\mathbf{Q}$-a.e. $(\omega, x) \in U_\varepsilon$ and $n > n_0(x)$,

(1) $\beta_x^{\omega,\xi_i}\left(B^{\Pi_j}(x, \exp(-n(N\chi_i - 2\varepsilon)))\right) > \exp\left(-n(N\chi_i - 2\varepsilon)(\overline{\gamma}_i + \varepsilon)\right)$; (by (3.31))
(2) $\beta_x^{\omega,\xi_i}\left(\mathcal{C}_0^{n-1}(x)\right) < \exp\left(-n(Nh_i - \varepsilon)\right)$; (by Lemma 3.8)
(3) $\beta_x^{\omega,\xi_{i-1}}\left(\mathcal{C}_0^{n-1}(x)\right) > \exp\left(-n(Nh_{i-1} + \varepsilon)\right)$; (by Lemma 3.8)
(4) $\xi_{i-1}(x) \cap \mathcal{C}_0^{n-1}(x) \subset B^{\Pi_j}(x, \exp(-n(N\chi_i - 2\varepsilon)))$. (by Lemma 3.5(iii)).

Take N_0 such that

$$\Delta := \{x \in U_\varepsilon \colon n_0(x) \leq N_0\}$$

satisfies $\mathbf{Q}(\Delta) > 0$. By (3.7) there exists $\widetilde{\Omega} \in \Omega$ with $\mathbf{P}(\widetilde{\Omega}) > 0$ such that for each $\omega \in \widetilde{\Omega}$ there exists $X^\omega \subset \Lambda^{\mathbb{N}}$ satisfying $\{\omega\} \times X^\omega \subset \Delta$ and $\beta^\omega(X^\omega) > 0$. Lemma 3.12 implies that for some $c > 0$ and each $\omega \in \widetilde{\Omega}$, there exists $Y^\omega \subset X^\omega$ with $\beta^\omega(Y^\omega) > 0$ such that for $x \in Y^\omega$ there exists $n = n(\omega, x) \geq N_0$ satisfying,

(5) $\beta_x^{\omega,\xi_i}\left(L \cap X^\omega\right) > c\beta_x^{\omega,\xi_i}(L)$, where $L = B^{\Pi_j}(x, \exp(-n(N\chi_i - 2\varepsilon)))$;
(6) $\beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_j}(x, 2\exp(-n(N\chi_i - 2\varepsilon)))\right) < \exp\left(-n(N\chi_i - 2\varepsilon)(\overline{\gamma}_{i-1} - \varepsilon)\right)$; (by (3.31))
(7) $\log(1/c) < n\varepsilon$.

Take $\omega \in \widetilde{\Omega}$ and $x \in Y^\omega$ such that (1)–(7) are satisfied with $n = n(\omega, x)$. By (5) and (1),

$$\beta_x^{\omega,\xi_i}\left(L \cap X^\omega\right) \geq c\beta_x^{\omega,\xi_i}(L) \geq c\exp(-n(N\chi_i\overline{\gamma}_i + O(\varepsilon))).$$

For each $I \in \mathcal{C}_0^{n-1}$ with $I \cap \xi_i(x) \cap L \cap X^\omega \neq \emptyset$, there is $y \in X^\omega$ such that $I = \mathcal{C}_0^{n-1}(y)$ and $\xi_i(y) = \xi_i(x)$. Thus, (2) implies

$$\beta_x^{\omega,\xi_i}(I) = \beta_y^{\omega,\xi_i}\left(\mathcal{C}_0^{n-1}(y)\right) < \exp(-n(Nh_i - \varepsilon)).$$

Hence, by $\xi_i(x) \subset \xi_{i-1}(x)$, combining the previous two equations gives

$$\begin{aligned}
\#\Big\{I \in \mathcal{C}_0^{n-1} \colon I \cap \xi_{i-1}(x) \cap L \cap X^\omega \neq \emptyset\Big\} \\
\geq \#\Big\{I \in \mathcal{C}_0^{n-1} \colon I \cap \xi_i(x) \cap L \cap X^\omega \neq \emptyset\Big\} \\
\geq c\exp\left(n(N(h_i - \chi_i\overline{\gamma}_i) - O(\varepsilon))\right).
\end{aligned}$$

On the other hand, for each $I \in \mathcal{C}_0^{n-1}$ with $I \cap \xi_{i-1}(x) \cap L \cap X^\omega \neq \emptyset$, there exists $z \in I \cap \xi_{i-1}(x) \cap L \cap X^\omega$. Thus,

$$\begin{aligned}
\xi_{i-1}(x) \cap I &= \xi_{i-1}(z) \cap \mathcal{C}_0^{n-1}(z) \\
&\subset B^{\Pi_j}(z, \exp(-n(N\chi_i - 2\varepsilon))) && \text{(by (4))} \\
&\subset B^{\Pi_j}(x, 2\exp(-n(N\chi_i - 2\varepsilon))). && \text{(by } z \in L)
\end{aligned}$$

It follows from (3) that

$$\beta_x^{\omega,\xi_{i-1}}(I) = \beta_z^{\omega,\xi_{i-1}}\left(\mathcal{C}_0^{n-1}(z)\right) \geq \exp(-n(Nh_{i-1}+\varepsilon)).$$

Hence

$$\begin{aligned}
&\beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_j}(x, 2\exp(-n(N\chi_i - 2\varepsilon)))\right) \\
&\quad \geq \#\left\{I \in \mathcal{C}_0^{n-1} \colon I \cap \xi_{i-1}(x) \cap L \cap X^\omega \neq \emptyset\right\} \exp(-n(Nh_{i-1}+\varepsilon)) \\
&\quad \geq \exp\left(\log c + n\left(N(h_i - h_{i-1} - \chi_i\overline{\gamma_i}) - O(\varepsilon)\right)\right).
\end{aligned}$$

From this, (6) and (7) it follows that

$$-N\chi_i\overline{\gamma}_{i-1} + O(\varepsilon) \geq N(h_i - h_{i-1} - \chi_i\overline{\gamma}_i) - O(\varepsilon).$$

Letting $\varepsilon \to 0$ and dividing by N give $h_{i-1} - h_i \geq \chi_i(\overline{\gamma}_{i-1} - \overline{\gamma}_i)$, a contradiction to (3.37). □

Proof of (D3). Suppose on the contrary that (D3) is not true. Then there exists $1 \leq i \leq j$ and $U \subset \Omega \times \Lambda^{\mathbb{N}}$ with $\mathbf{Q}(U) > 0$ such that for $(\omega, x) \in U$,

$$\underline{\gamma}_{i,j}^\omega(x) + \vartheta_i > \underline{\gamma}_{i-1,j}^\omega(x).$$

Then there exist $\alpha > 0$ and real numbers $\underline{\gamma}_{i-1}, \underline{\gamma}_{i-1}$ such that

$$\underline{\gamma}_i + \vartheta_i > \underline{\gamma}_{i-1} + \alpha, \tag{3.39}$$

and for every $\varepsilon > 0$, there exists $U_\varepsilon \subset U$ with $\mathbf{Q}(U_\varepsilon) > 0$ so that for $(\omega, x) \in U_\varepsilon$,

$$\left|\underline{\gamma}_{i-1,j}^\omega(x) - \underline{\gamma}_{i-1}\right| < \varepsilon/2 \quad \text{and} \quad \left|\underline{\gamma}_{i,j}^\omega(x) - \underline{\gamma}_i\right| < \varepsilon/2. \tag{3.40}$$

Let $0 < \varepsilon < \alpha/4$. By Egorov's theorem, there exists $\Delta \subset U_\varepsilon$ with $\mathbf{Q}(\Delta) > 0$ and $N_0 \in \mathbb{N}$ such that for $(\omega, x) \in \Delta$ and $n > N_0$,

$$\beta_x^{\omega,\xi_i}\left(B^{\Pi_j}(x, 2\exp(-n))\right) \leq \exp\left(-n\left(\underline{\gamma}_i - \varepsilon\right)\right). \tag{3.41}$$

By (3.7), there exists $\widetilde{\Omega} \in \Omega$ with $\mathbf{P}(\widetilde{\Omega}) > 0$ so that for each $\omega \in \widetilde{\Omega}$ there exists $X^\omega \subset \Lambda^{\mathbb{N}}$ satisfying $\{\omega\} \times X^\omega \subset \Delta$ and $\beta^\omega(X^\omega) > 0$. Lemma 3.12 implies that for some $c > 0$ and each $\omega \in \widetilde{\Omega}$, there exists $Y^\omega \subset X^\omega$ with $\beta^\omega(Y^\omega) > 0$ such that for $x \in Y^\omega$ there exists $N_1 \geq N_0$ so that for $\omega \in \widetilde{\Omega}$, $x \in Y^\omega$ and $n \geq N_1$,

$$\beta_x^{\omega,\xi_{i-1}}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) > c\beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_j}(x, \exp(-n))\right). \tag{3.42}$$

Then

$$\begin{aligned}
&\beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_j}(x, \exp(-n))\right) \\
&\quad \leq c^{-1}\beta_x^{\omega,\xi_{i-1}}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) \\
&\quad \leq c^{-1}\int_{\Lambda^{\mathbb{N}}} \beta_y^{\omega,\xi_i}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) \mathrm{d}\beta_x^{\omega,\xi_{i-1}}(y) \qquad (\text{by } \xi_{i-1} \prec \xi_i) \\
&\quad \leq c^{-1}\int_{B^{\Pi_i}(x, \exp(-n))} \beta_y^{\omega,\xi_i}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) \mathrm{d}\beta_x^{\omega,\xi_{i-1}}(y),
\end{aligned} \tag{3.43}$$

where the last inequality holds since combining $y \in \xi_{i-1}(x)$ and $\xi_i(y) \cap X^\omega \cap B^{\Pi_j}(x, \exp(-n)) \neq \emptyset$ implies $y \in B^{\Pi_i}(x, \exp(-n))$. To see that, take $z \in \xi_i(y) \cap X^\omega \cap B^{\Pi_j}(x, \exp(-n))$. Since $\Pi_i(z) = \Pi_i(y)$ and $\pi_{[i]}\Pi_j = \Pi_i$ by $i \le j$, we have

$$\|\Pi_i(y) - \Pi_i(x)\| = \|\Pi_i(z)) - \Pi_i(x)\| \le \|\Pi_j(z) - \Pi_j(x)\| \le \exp(-n),$$

which implies $y \in B^{\Pi_i}(x, \exp(-n))$. Moreover, it follows from $z \in B^{\Pi_j}(x, \exp(-n))$ that

$$B^{\Pi_j}(x, \exp(-n)) \subset B^{\Pi_j}(z, 2\exp(-n)).$$

Hence,

$$\begin{aligned} \beta_y^{\omega,\xi_i}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) &= \beta_z^{\omega,\xi_i}\left(X^\omega \cap B^{\Pi_j}(x, \exp(-n))\right) && (\text{by } \xi_i(z) = \xi_i(y)) \\ &\le \beta_z^{\omega,\xi_i}\left(B^{\Pi_j}(z, 2\exp(-n))\right) \\ &\le \exp\Big(-n(\underline{\gamma}_i - \varepsilon)\Big). && (\text{by (3.41) and } z \in X^\omega) \end{aligned}$$

Combining this with (3.43) shows that for $\omega \in \widetilde{\Omega}$ and $x \in Y^\omega$,

$$\beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_j}(x, \exp(-n))\right) \le \exp\Big(-\log c - n(\underline{\gamma}_i - \varepsilon)\Big)\, \beta_x^{\omega,\xi_{i-1}}\left(B^{\Pi_i}(x, \exp(-n))\right).$$

By taking logarithm, dividing by n and letting $n \to \infty$, we have $\underline{\gamma}_{i-1,j}^{\omega}(x) \ge \underline{\gamma}_i - \varepsilon + \vartheta_i$. Then applying (3.40) shows

$$\underline{\gamma}_{i-1} \ge \underline{\gamma}_i + \vartheta_i - 2\varepsilon.$$

Letting $\varepsilon \to 0$ gives $\underline{\gamma}_{i-1} \ge \underline{\gamma}_i + \vartheta_i$, a contradiction to (3.39). □

4. The disintegrations with respect to linear parts

In this and all the subsequent sections, we fix $N \in \mathbb{N}$ and let Γ be a partition of $\Lambda^{\mathbb{N}}$ so that for $x, y \in \Lambda^{\mathbb{N}}$, $x|N = y|N$ implies $\Gamma(x) = \Gamma(y)$, which in turn implies $A_{\varphi_{x|N}} = A_{\varphi_{y|N}}$. Specifically,

$$L \prec \Gamma \prec \{[I] \colon I \in \Lambda^N\}, \tag{4.1}$$

where L is the partition of $\Lambda^{\mathbb{N}}$ defined by $L(x) = L(y)$ if and only if $A_{\varphi_{x|N}} = A_{\varphi_{y|N}}$ for $x, y \in \Lambda^{\mathbb{N}}$. We set $T = \sigma^N$ and $\mathcal{A} = \vee_{i=0}^{\infty} T^{-i}\Gamma$. Recall the definitions of $\Omega, \mathbf{P}, \beta^\omega, \mu^\omega$ from Section 2.4. In this section we introduce some properties of $\mathcal{A}$ and the associated random measures.

We begin with some notations. For $\omega \in \Omega$, where $\omega = \mathcal{A}(x)$ with $x \in \Lambda^{\mathbb{N}}$, and $n \ge 0$, define

$$A^{\omega|n} := A_{\varphi_{x|nN}} \quad \text{and} \quad A^{-\omega|n} := (A^{\omega|n})^{-1}.$$

This is well defined since, by (4.1), it is independent of the choice of x. For $1 \le j \le d$, let the j-th entry on the diagonal of $A^{\omega|n}$ be denoted by $A_j^{\omega|n}$. Define

$$\lambda_j^{\omega|n} := \left|A_j^{\omega|n}\right| \quad \text{and} \quad \chi_j^{\omega|n} := -\log \lambda_j^{\omega|n}.$$

Let $r_{\min} := \min\{|r_{i,j}| \colon 1 \le i, j \le d\}$ and $r_{\max} := \max\{|r_{i,j}| \colon 1 \le i, j \le d\}$. Then

$$r_{\min}^{Nn} \le \lambda_j^{\omega|n} \le r_{\max}^{Nn} \quad \text{for } 1 \le j \le d. \tag{4.2}$$

The following lemma is a direct consequence of Lemma 3.4, (3.7) and Egorov's theorem.

Lemma 4.1. *For $\eta \in (0,1)$ there exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1 - \eta$ so that for $\omega \in \overline{\Omega}$ and $n \in \mathbb{N}$ with $\eta^{-1} \ll n$, we have $\left|\chi_j^{\omega|n} - nN\chi_j\right| < n\eta$ for $1 \le j \le d$.*

The random measure μ^ω exhibits a convolution structure. For $\omega \in \Omega$ and $n \geq 0$, define

$$\nu_n^\omega = \sum_{u \in \Lambda^{nN}} \beta^\omega([u]) \delta_{\varphi_u(0)}.$$

Since $A_{\varphi_u} = A^{\omega|n}$ for $u \in \Lambda^{nN}$ with $\beta^\omega([u]) \neq 0$, it follows from (2.13) that

$$\mu^\omega = \nu_n^\omega * A^{\omega|n} \mu^{T^n \omega}, \tag{4.3}$$

where $A\theta$ denotes the pushforward of a measure θ by a matrix A.

4.1. **Nonconformal partition.** Fix $\omega \in \Omega$. Following [46], we define the nonconformal partitions used to analyze the entropy growth of μ^ω. For $n \in \mathbb{Z}$, let $\mathcal{D}_n^d$ be the n-th level dyadic partition of $\mathbb{R}^d$, that is,

$$\mathcal{D}_n^d = \left\{ \frac{k}{2^n} + \left[0, \frac{1}{2^n}\right)^d : k \in \mathbb{Z}^d \right\}.$$

For $t \in \mathbb{R}$, define $\mathcal{D}_t^d = \mathcal{D}_{\lfloor t \rfloor}^d$. We omit the superscript d when the ambient space is clear from the context. For $\omega \in \Omega$ and $n \geq 0$, define

$$\mathcal{E}_n^\omega := A^{\omega|n} \mathcal{D}_0^d = \{A^{\omega|n} D \colon D \in \mathcal{D}_0^d\} = \bigtimes_{j=1}^d \lambda_j^{\omega|n} \mathcal{D}_0^1. \tag{4.4}$$

It follows that

$$A^{\omega|b} \pi_J^{-1} \mathcal{E}_n^{T^b \omega} = \pi_J^{-1} \mathcal{E}_{n+b}^\omega \quad \text{for } b \geq 0 \text{ and } J \subset [d], \tag{4.5}$$

and

$$\mathcal{E}_n^\omega \text{ and } \bigtimes_{j=1}^d \mathcal{D}^1_{\chi_j^{\omega|n}} \text{ are } O(1)\text{-commensurable.} \tag{4.6}$$

For $y \in \mathbb{R}^d$, we define the translation map $T_y(x) = x + y$, $x \in \mathbb{R}^d$. It is readily checked that

$$\pi_J \mathcal{E}_n^\omega \text{ and } T_y^{-1} \pi_J^{-1} \mathcal{E}_n^\omega \text{ are } O(1)\text{-commensurable for } J \subset [d] \text{ and } y \in \mathbb{R}^d. \tag{4.7}$$

Next, suppose f, g are two maps from a set X to $\mathbb{R}^d$ such that for some $C > 1$,

$$|\pi_j(f(x) - g(x))| \leq C \lambda_j^{\omega|n} \quad \text{for } 1 \leq j \leq d \text{ and } x \in X.$$

Then

$$f^{-1} \pi_J^{-1} \mathcal{E}_n^\omega \text{ and } g^{-1} \pi_J^{-1} \mathcal{E}_n^\omega \text{ are } O\left(C^d\right)\text{-commensurable for } J \subset [d]. \tag{4.8}$$

Combining (4.3), Lemma 2.2(iii) and (4.7), we obtain the following inequality for $m, n \geq 0$,

$$H\left(\mu^\omega, \mathcal{E}_{m+n}^\omega \mid \mathcal{E}_n^\omega\right) \geq H\left(\mu^{T^n \omega}, \mathcal{E}_m^{T^n \omega}\right) - O(1)\,. \tag{4.9}$$

This estimate is the major advantage of considering μ^ω and $\mathcal{E}_n^\omega$.

4.2. **Component measure.** Fix $\omega \in \Omega$. We introduce the component measures along $\mathcal{E}_n^\omega$. Given $\theta \in \mathcal{M}(\mathbb{R}^d)$ and $n \geq 0$, let $\theta_{x,n}^\omega$ be a measure-valued random element such that $\theta_{x,n}^\omega = \theta_{\mathcal{E}_n^\omega(x)}$ with probability $\theta(\mathcal{E}_n(x))$ for $x \in \mathbb{R}^d$. Thus, for a event $\mathcal{U} \subset \mathcal{M}(\mathbb{R}^d)$,

$$\mathbb{P}\{\theta_{x,n}^\omega \in \mathcal{U}\} = \theta\left\{x \in \mathbb{R}^d \colon \theta_{\mathcal{E}_n^\omega(x)} \in \mathcal{U}\right\}.$$

We call $\theta_{x,n}^\omega$ an *n-th level component* of θ given $\omega \in \Omega$ and $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ with $\theta(\mathcal{E}_n^\omega(x)) > 0$, we write $\theta_{x,n}^\omega$ in place of $\theta_{\mathcal{E}_n^\omega(x)}$ even when no randomness is involved. Thus, for $n \geq 0$,

$$\theta = \int \theta_{x,n}^\omega \, \mathrm{d}\theta(x). \tag{4.10}$$

We can also choose a random scale n uniformly from a range. For example, for a finite set $I \subset \mathbb{N}$, define

$$\mathbb{P}_{i\in I}\{\theta_{x,i}^\omega \in \mathcal{U}\} := \frac{1}{|I|}\sum_{i\in I}\mathbb{P}\{\theta_{x,i}^\omega \in \mathcal{U}\}.$$

Let $\mathbb{E}$ and $\mathbb{E}_{i\in I}$ denote the corresponding expectation with respect to $\mathbb{P}$ and $\mathbb{P}_{i\in I}$. Thus, for each bounded measurable function $f \colon \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}$ and $n \geq 0$,

$$\mathbb{E}_{i=n}\big(f(\theta_{x,i}^\omega)\big) = \int f(\theta_{\mathcal{E}_n^\omega(x)})\, \mathrm{d}\theta(x).$$

In particular, for $k, n \geq 0$,

$$H\big(\theta, \mathcal{E}_{n+k}^\omega \mid \mathcal{E}_n^\omega\big) = \mathbb{E}_{i=n}\big(H\big(\theta_{x,i}^\omega, \mathcal{E}_{n+k}^\omega\big)\big). \tag{4.11}$$

We finish this section with the a useful lemma relating the entropies of a measure and its components. The proof is almost identical to [26, Lemma 3.4] and is therefore omitted.

Lemma 4.2. *Let $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ with $\operatorname{diam}(\operatorname{supp}\theta) \leq R$ for some $R \geq 1$. Then for all $\omega \in \Omega$ and every $1 \leq m \leq n$,*

$$\begin{aligned}\frac{1}{n}H(\theta, \mathcal{E}_n^\omega) &= \mathbb{E}_{1\leq q\leq n}\left(\frac{1}{m}H\big(\theta_{x,q}^\omega, \mathcal{E}_{q+m}^\omega\big)\right) + O\left(\frac{m+\log R}{n}\right)\\ &= \mathbb{E}_{1\leq q\leq n}\left(\frac{1}{m}H\big(\theta, \mathcal{E}_{q+m}^\omega \mid \mathcal{E}_q^\omega\big)\right) + O\left(\frac{m+\log R}{n}\right).\end{aligned}$$

5. Entropy of repeated self-convolutions

This section is devoted to proving the following proposition, which is analogous to [46, Proposition 1.15] for the random measures. It plays a crucial role in establishing the entropy increase result. The proof is adapted from [46]. To account for the dependence on ω and other additional parameters, based on the dynamics on $(\Omega, \mathbf{P})$ we adapt the arguments to prove the modified version of the statements. For clarity, we provide the necessary details.

Proposition 5.1. *For $\varepsilon \in (0,1)$, there is $\delta > 0$ so that the following holds. Let $\eta \in (0,1)$ and $m_1, \ldots, m_d, k_1, \ldots, k_d \in \mathbb{N}$ be with $\varepsilon^{-1} \ll \eta^{-1} \ll m_d \ll k_d \ll m_{d-1} \ll \cdots \ll k_2 \ll m_1 \ll k_1$. There exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) \geq 1 - \eta$ so that for $n \in \mathbb{N}$ with $k_1 \ll n$ and $\omega \in \overline{\Omega}$ the following holds. Let $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ with $\operatorname{diam}(\operatorname{supp}\theta) \leq \varepsilon^{-1}$ and $\frac{1}{n}H(\theta, \mathcal{E}_n^\omega) > \varepsilon$. Then there exist $j \in [d]$ and $Q^\omega \subset [n]$ with $\#_n(Q^\omega) \geq \delta$ so that*

$$\frac{1}{m_j}H\left(\theta^{*k_j}, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega \vee \pi_{[d]\setminus\{j\}}^{-1}\mathcal{E}_{q+m_j}^\omega\right) > N\chi_j - \varepsilon \quad \textit{for } q \in Q^\omega. \tag{5.1}$$

5.1. **Entropy of self-convolutions under a condition on variance.** The purpose of this subsection is to prove the following lemma, which is analogous to [46, Lemma 3.2].

Lemma 5.2. *Let $\eta \in (0,1)$ and $m, \ell, k \in \mathbb{N}$ be with $\eta^{-1} \ll m \ll \ell \ll k$. There exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1-\eta$ so that, for $n \in \mathbb{N}$ with $k \ll n$ and $\omega \in \overline{\Omega}$, there is $B^\omega \subset [n]$ with $\#_n(B^\omega) > 1-\eta$ so that the following holds. Let $\theta_1, \dots, \theta_k \in \mathcal{M}_c(\mathbb{R}^d)$ be with $\operatorname{diam}(\operatorname{supp}\theta_i) \le \eta^{-1}$ for $1 \le i \le k$. Set $\rho := \theta_1 * \cdots * \theta_k$. Suppose that there exists $1 \le j \le d$ so that $\operatorname{Var}(\pi_j \rho) \ge \eta k$ and $\operatorname{Var}(\pi_{j'}\rho) \le \eta^{-1}$ for $1 \le j' < j$. Then setting $a := \lfloor \log k/(2N\chi_j) \rfloor$, we have for $\omega \in \overline{\Omega}$,*

$$\frac{1}{m} H\Big(\rho, \mathcal{E}^{T^b\omega}_{\ell-a+m} \mid \mathcal{E}^{T^b\omega}_{\ell-a} \vee \pi^{-1}_{[d]\setminus\{j\}} \mathcal{E}^{T^b\omega}_{\ell-a+m}\Big) > N\chi_j - \eta \quad \textit{for } b \in B^\omega.$$

Proof. The proof is adapted from [46, Lemma 3.2]. To account for the dependence on additional parameters, we include the details for clarity. For $1 \le j \le d$, the coordinate map from $\mathbb{R}^d$ to $\mathbb{R}$ is denoted as $\widetilde{\pi}_j(x) = \langle x, e_j \rangle$ for $x \in \mathbb{R}^d$. After a translation of ρ, by Lemma 2.1(iv) we can assume that the mean $\langle \widetilde{\pi}_j \rho \rangle = 0$ for $1 \le j' \le d$ and $\operatorname{supp}\theta_i \subset [-\eta, \eta]^d$ for $1 \le i \le k$.

Let $\varepsilon \in (0,1)$ be with $\eta^{-1} \ll \varepsilon^{-1} \ll m$. By Lemma 4.1 and the T-invariance of $\mathbf{P}$, there exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1 - \varepsilon/2$ so that for $\omega \in \overline{\Omega}$ and $1 \le j' \le d$,

$$\left| \frac{\chi^{\omega|a}_{j'}}{\log k} - \frac{\chi_{j'}}{2\chi_j} \right| < \varepsilon, \qquad \left| \chi^{\omega|(\ell+m)}_{j'} - (\ell+m)N\chi_{j'} \right| < \varepsilon, \tag{5.2}$$

and

$$\left| \chi^{T^\ell\omega|m}_{j} - mN\chi_j \right| < m\varepsilon. \tag{5.3}$$

In what follows we take $\omega \in \overline{\Omega}$.

We first show that

$$\frac{1}{m} H\Big(\pi_j A^{\omega|a}\rho, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) \ge N\chi_j - \frac{\eta}{4}, \tag{5.4}$$

where $\mathcal{C}^\omega := \mathcal{E}^\omega_\ell \vee \pi^{-1}_{[d]\setminus\{j\}} \mathcal{E}^\omega_{\ell+m}$. The proof of (5.4) is based on the Berry-Essen theorem. Next, we estimate the moments of corresponding measures. For $1 \le i \le k$ and $s = 2, 3$, it follows from (5.2) that

$$\int |t|^s \, \mathrm{d}\widetilde{\pi}_j A^{\omega|a}\theta_i(t) = \exp\Big(-s\chi^{\omega|a}_j\Big) \int |t|^s \, \mathrm{d}\widetilde{\pi}_j \theta_i(t) = O\Big(\eta^{-s} k^{-s/2+s\varepsilon}\Big). \tag{5.5}$$

Thus, the variance satisfies

$$\operatorname{Var}(\widetilde{\pi}_j A^{\omega|a}\rho) = \sum_{i=1}^k \operatorname{Var}(\widetilde{\pi}_j A^{\omega|a}\theta_i) = O\big(\eta^{-2}k^{2\varepsilon}\big).$$

Moreover,

$$\operatorname{Var}(\widetilde{\pi}_j A^{\omega|a}\rho) = \exp\Big(-2\chi^{\omega|a}_j\Big) \operatorname{Var}(\widetilde{\pi}_j \rho) \ge \eta k^{-2\varepsilon}.$$

Hence

$$\frac{\sum_{i=1}^k \int |t|^3 \, \mathrm{d}\widetilde{\pi}_j A^{\omega|a}\theta_i(t)}{\operatorname{Var}(\widetilde{\pi}_j A^{\omega|a}\rho)^{3/2}} = O\Big(\eta^{-9/2} k^{-1/2+6\varepsilon}\Big).$$

Combining all above with $\varepsilon^{-1} \ll m \ll \ell$ and [46, Theorem 3.1 and Lemma 3.3], we conclude from Lemma 2.2(iv) and (5.3) that

$$\begin{aligned}
&\frac{1}{mN\chi_j} H\Big(\widetilde{\pi}_j A^{\omega|a}\rho, \mathcal{D}^1_{\chi_j^{\omega|\ell}+\chi_j^{T^\ell\omega|m}} \mid \mathcal{D}^1_{\chi_j^{\omega|\ell}}\Big) \\
&\quad\geq \frac{1}{mN\chi_j} H\Big(\widetilde{\pi}_j A^{\omega|a}\rho, \mathcal{D}^1_{\chi_j^{\omega|\ell}+mN\chi_j} \mid \mathcal{D}^1_{\chi_j^{\omega|\ell}}\Big) - O(\varepsilon) \\
&\quad\geq 1 - \varepsilon - O(\varepsilon) \geq 1 - O(\varepsilon)\,.
\end{aligned}$$

By (4.6) and $\eta^{-1} \ll \varepsilon^{-1}$, this proves (5.4).

We proceed to estimate the error caused by π_j in (5.4). For $j' \in [d]\setminus\{j\}$, set

$$S_{j'} := \Big\{x \in \mathbb{R}^d\colon \Big|\pi_{j'}A^{\omega|a}x\Big| \leq \exp\left(-2N\chi_d(\ell+m)\right)\Big\}\,,$$

and define $S := \cap_{j'\in[d]\setminus\{j\}} S_{j'}$. For $x \in S$,

$$\Big|A^{\omega|a}x - \pi_j A^{\omega|a}x\Big| = O(\exp\left(-2N\chi_d(\ell+m)\right))\,.$$

Hence by (5.2) and (4.8),

$$H\Big(A^{\omega|a}\rho_S, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) = H\Big(\pi_j A^{\omega|a}\rho_S, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) + O(1). \tag{5.6}$$

For $j < j' \leq d$, it follows from (5.2) that

$$\mathrm{Var}(\widetilde{\pi}_{j'}A^{\omega|a}\rho) = \exp\left(-2\chi_{j'}^{\omega|a}\right)\sum_{i=1}^k \mathrm{Var}(\widetilde{\pi}_{j'}\theta_i) = O\Big(\eta^{-2}k^{1-\chi_{j'}/\chi_j+2\varepsilon}\Big)\,.$$

For $1 \leq j' < j$, it follows from $\mathrm{Var}(\pi_{j'}\rho) \leq \eta^{-1}$ and (5.2) that

$$\mathrm{Var}(\widetilde{\pi}_{j'}A^{\omega|a}\rho) \leq \eta^{-1}\exp\left(-2\chi_{j'}^{\omega|a}\right) = O\left(\eta^{-1}k^{-\chi_{j'}/\chi_j+2\varepsilon}\right)\,.$$

Recall that $\chi_1 < \cdots < \chi_d$. By $\eta^{-1} \ll \varepsilon^{-1}$, there is $\delta > 0$ only depending on $\chi_1, \ldots, \chi_d$ so that

$$\mathrm{Var}(\widetilde{\pi}_{j'}A^{\omega|a}\rho) = O(\eta^{-2}k^{-\delta}) \quad \text{for } j' \in [d]\setminus\{j\}.$$

From this, since the mean $\langle\widetilde{\pi}_{j'}\rho\rangle = 0$ for $j' \in [d]$, and by Chebyshev's inequality,

$$\begin{aligned}
\rho(S^c) &\leq \sum_{j'\in[d]\setminus\{j\}} \rho(S^c_{j'}) \leq \sum_{j'\in[d]\setminus\{j\}} \exp\left(4N\chi_d(\ell+m)\right)\mathrm{Var}(\widetilde{\pi}_{j'}A^{\omega|a}\rho) \\
&= O\Big(\exp\left(4N\chi_d(\ell+m)\right)\eta^{-2}k^{-\delta}\Big)\,.
\end{aligned} \tag{5.7}$$

By $\operatorname{supp}\pi_j A^{\omega|a}\rho \subset [-k\eta^{-1}, k\eta^{-1}]^d$ and (5.2),

$$H\Big(\pi_j A^{\omega|a}\rho_{S^c}, \mathcal{E}^\omega_{\ell+m}\Big) = O(\ell + m + \log(k\eta^{-1})).$$

From the above two equations, it follows from $\eta^{-1} \ll m \ll \ell \ll k$ that

$$\frac{\rho(S^c)}{m} H\Big(\pi_j A^{\omega|a}\rho_{S^c}, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) \leq \frac{\eta}{4}. \tag{5.8}$$

Hence

$$\begin{aligned}
&\frac{1}{m}H\Big(A^{\omega|a}\rho, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) \\
&\quad\geq \frac{\rho(S)}{m}H\Big(A^{\omega|a}\rho_S, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) \qquad \text{(by Lemma 2.2(iii))}
\end{aligned}$$

$$\geq \frac{\rho(S)}{m} H\Big(\pi_j A^{\omega|a} \rho_S, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) - O\Big(\frac{1}{m}\Big) \qquad \text{(by (5.6))}$$

$$\geq \frac{\rho(S)}{m} H\Big(\pi_j A^{\omega|a} \rho_S, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) - \frac{\eta}{4} \qquad \text{(by } \eta^{-1} \ll m\text{)}$$

$$+ \frac{\rho(S^c)}{m} H\Big(\pi_j A^{\omega|a} \rho_{S^c}, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) - \frac{\eta}{4} \qquad \text{(by (5.8))}$$

$$\geq \frac{1}{m} H\Big(\pi_j A^{\omega|a} \rho, \mathcal{E}^\omega_{\ell+m} \mid \mathcal{C}^\omega\Big) - \frac{3}{4}\eta \qquad \text{(by Lemma 2.2(iii) and (5.7))}$$

$$\geq N\chi_j - \eta. \qquad \text{(by (5.4))}$$

By (4.4) and (4.5), this implies that

$$\frac{1}{m} H\Big(\rho, \mathcal{E}^{T^a\omega}_{\ell-a+m} \mid \mathcal{E}^{T^a\omega}_{\ell-a} \vee \pi^{-1}_{[d]\setminus\{j\}} \mathcal{E}^{T^a\omega}_{\ell-a+m}\Big) \geq N\chi_j - \eta.$$

Since $\mathbf{P}(\overline{\Omega}) > 1 - \varepsilon/2 > 1 - \eta/2$ and $a = O(\log k) \ll n$, the proof is finished by applying Birkhoff's ergodic theorem and Egorov's theorem to $\mathbf{1}_{\overline{\Omega}}$. □

5.2. **Positive entropy implies nonnegligible variance.** Based on Chebyshev's inequality and (4.2), the proof of the next lemma is almost identical to [26, Lemma 4.4] and so omitted.

Lemma 5.3. *Let $\varepsilon, \delta \in (0,1)$ and $m \in \mathbb{N}$ be with $\varepsilon^{-1} \ll m \ll \delta^{-1}$. Let $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ such that $\operatorname{diam}(\operatorname{supp}\theta) \leq \varepsilon^{-1}$ and $\operatorname{Var}(\pi_j\theta) \leq \delta$ for each $1 \leq j \leq d$. Then $\frac{1}{m} H(\theta, \mathcal{E}^\omega_m) < \varepsilon$ for $\omega \in \Omega$.*

The following lemma is analogous to [46, Lemma 3.5], providing a nonnegligible proportion of components with positive variance based on the assumption of positive entropy. The proof is nearly identical to that of [46, Lemma 3.5], based on Lemma 5.3, and is therefore omitted.

Lemma 5.4. *For $\varepsilon \in (0,1)$ there exists $\delta > 0$ so that the following holds. Let $n \in \mathbb{N}$ be with $\varepsilon^{-1} \ll n$. Let $\omega \in \Omega$ and $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ be with $\operatorname{diam}(\operatorname{supp}\theta) \leq \varepsilon^{-1}$ and $\frac{1}{n} H(\theta, \mathcal{E}^\omega_n) > \varepsilon$. Then there exists $B^\omega \subset [n]$ with $\#_n(B^\omega) \geq \delta$ so that*

$$\mathbb{P}_{i=b}\Big\{\operatorname{Var}(\pi_j A^{-\omega|i} \theta^\omega_{x,i}) > \delta \text{ for some } 1 \leq j \leq d\Big\} \geq \delta \quad \text{for } b \in B^\omega.$$

5.3. **Proof of Proposition 5.1.** Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. The proof is adapted from [46, Proposition 1.15], with Lemmas 5.2 and 5.4 in roles of [46, Lemmas 3.2 and 3.5], respectively. To account for the dependence on additional parameters and for clarity, we include the necessary details.

Let $\delta \in (0,1)$ and $\ell_1, \ldots, \ell_d \in \mathbb{N}$ be with

$$\varepsilon^{-1} \ll \delta^{-1} \ll \eta^{-1} \ll m_d \ll \ell_d \ll k_d \ll m_{d-1} \ll \cdots \ll k_2 \ll m_1 \ll \ell_1 \ll k_1 \ll n. \tag{5.9}$$

Define $\widetilde{k_j} = \lfloor \delta k_j/(2d) \rfloor$ for $1 \leq j \leq d$. By $\ell_j \ll k_j$ and $\delta^{-1} \ll k_j$, we have $\ell_j \ll \widetilde{k_j}$. Let $\eta_d := \eta$ and $\eta_j := k^{-1}_{j+1}$ for $1 \leq j < d$. Then $\eta_j \leq \eta$ and $\eta_j^{-1} \ll m_j \ll \ell_j \ll \widetilde{k_j} \ll n$ for $1 \leq j \leq d$.

Let $\overline{\Omega}$ be the intersection of the $\overline{\Omega}$'s obtained by applying Lemma 5.2 repeatedly with η_j, m_j, ℓ_j, k_j in place of η, m, ℓ, k for $1 \leq j \leq d$. Note that $k_j \ll n$ for $1 \leq j \leq d$. For $\omega \in \overline{\Omega}$, let B^ω be the intersection of corresponding B^ω's obtained by applying Lemma 5.2 with n in place of n. Then $\mathbf{P}(\overline{\Omega}) > 1 - d\eta$ and for $\omega \in \overline{\Omega}$, $\#_n(B^\omega) > 1 - d\eta$. In what follows we take $\omega \in \overline{\Omega}$, and let $B^\omega \subset [n]$ accordingly. By Lemma 5.4 and $\varepsilon^{-1} \ll \delta^{-1} \ll \eta^{-1}$, there exists $\overline{B}^\omega \subset B^\omega$ with $\#_n(\overline{B}^\omega) > \delta - d\eta > \delta/2$ so that for $b \in \overline{B}^\omega$,

$$\mathbb{P}_{i=b}\Big\{\mathrm{Var}(\pi_j A^{-\omega|b}\theta^\omega_{x,i}) > \delta \text{ for some } 1 \le j \le d\Big\} > \delta.$$

For $1 \le j \le d$, let B^ω_j be the set of all $b \in \overline{B}^\omega$ so that

$$\mathbb{P}_{i=b}\Big\{\mathrm{Var}(\pi_j A^{-\omega|i}\theta^\omega_{x,i}) > \eta_j \text{ and } \mathrm{Var}(\pi_{j'} A^{-\omega|i}\theta^\omega_{x,i}) \le \eta_{j'} \text{ for } 1 \le j' < j\Big\} > \delta/d.$$

It is clear that $\overline{B}^\omega \subset \cup_{j=1}^d B^\omega_j$. Since $\#_n(\overline{B}^\omega) > \delta/2$, it follows that $\#_n(B^\omega_j) > \delta/(2d)$ for some $1 \le j \le d$. Fix such j until the end of the proof.

Note that

$$\varepsilon^{-1} \ll \delta^{-1} \ll \eta_j^{-1} \ll m_j \ll \ell_j \ll k_j \ll n \quad \text{ and } \quad \eta_{j'} \le k_j^{-1} \text{ for } 1 \le j' < j.$$

Let $b \in B^\omega_j$ be given, and define

$$Y := \{x \in \mathbb{R}^d\colon \mathrm{Var}(\pi_j A^{-\omega|b}\theta^\omega_{x,b}) > \eta_j \text{ and } \mathrm{Var}(\pi_{j'} A^{-\omega|b}\theta^\omega_{x,b}) \le \eta_{j'} \text{ for } 1 \le j' < j\}.$$

Recall $\widetilde{k}_j = \lfloor \delta k_j/(2d)\rfloor$, and write $k = \widetilde{k}_j$ for short. Set

$$Z := \Big\{(x_1, \dots, x_{k_j}) \in (\mathbb{R}^d)^{k_j} \colon \#\{1 \le s \le k_j \colon x_s \in Y\} \ge k\Big\}.$$

Since $\theta(Y) > \delta/d$ and $\delta^{-1} \ll k_j$, the weak law of large numbers implies $\theta^{\times k_j}(Z) > 1 - \delta$.

Let $(x_1, \dots, x_{k_j}) \in Z$ be given. Then there exist integers $1 \le s_1 < \dots < s_k \le k_j$ so that $x_{s_i} \in Y$ for $1 \le i \le k$. Note that

$$\mathrm{diam}\left(\mathrm{supp}\, A^{-\omega|b}\theta^\omega_{x_{s_i},b}\right) = O(1) \quad \text{for } 1 \le i \le k.$$

Set

$$\rho := A^{-\omega|b}\theta^\omega_{x_{s_1},b} * \dots * A^{-\omega|b}\theta^\omega_{x_{s_k},b}.$$

We have

$$\mathrm{Var}(\pi_j\rho) = \sum_{i=1}^k \mathrm{Var}(\pi_j A^{-\omega|b}\theta^\omega_{x_{s_i},b}) \ge k\eta_j,$$

and for each $1 \le j' < j$, recalling $k = \widetilde{k}_j = \lfloor \delta k_j/(2d)\rfloor$,

$$\mathrm{Var}(\pi_{j'}\rho) = \sum_{i=1}^k \mathrm{Var}(\pi_{j'} A^{-\omega|b}\theta^\omega_{x_{s_i},b}) \le k\eta_{j'} = O\big(\delta k_j \eta_{j'}\big) \le 1.$$

Recall the definition of B^ω and B^ω_j. Set $a := \lfloor \log k/(2N\chi_j)\rfloor$. It follows from Lemma 5.2 that

$$\frac{1}{m_j} H\Big(\rho, \mathcal{E}^{T^b\omega}_{\ell_j - a + m_j} \mid \mathcal{E}^{T^b\omega}_{\ell_j - a} \vee \pi^{-1}_{[d]\setminus\{j\}}\mathcal{E}^{T^b\omega}_{\ell_j - a + m_j}\Big) > N\chi_j - \delta \quad \text{for } b \in B^\omega_j. \tag{5.10}$$

For $s \in \mathbb{Z}$ and $b \ge 0$, write $\mathcal{C}^{T^b\omega}_s := \mathcal{E}^{T^b\omega}_{s+\ell_j-a} \vee \pi^{-1}_{[d]\setminus\{j\}}\mathcal{E}^{T^b\omega}_{s+\ell_j-a+m_j}$ for short. Since (5.10), $k \le k_j$ and $\delta^{-1} \ll m_j$, we conclude from (4.7) and the concavity of entropy that for $b \in B^\omega_j$,

$$\frac{1}{m_j} H\Big(*_{s=1}^{k_j} A^{-\omega|b}\theta^\omega_{x_s,b}, \mathcal{E}^{T^b\omega}_{\ell_j-a+m_j} \mid \mathcal{C}^{T^b\omega}_0\Big) > N\chi_j - 2\delta.$$

Then by (4.5),

$$\frac{1}{m_j} H\Big(*_{s=1}^{k_j} \theta^\omega_{x_s,b}, \mathcal{E}^{\omega}_{b+\ell_j-a+m_j} \mid \mathcal{C}^{\omega}_b\Big) > N\chi_j - 3\delta. \tag{5.11}$$

Note that by (4.10),

$$\theta^{*k_j} = \int *_{s=1}^{k_j} \theta^{\omega}_{x_s,b} \, \mathrm{d}\theta^{\times k_j}(x_1, \dots, x_{k_j}).$$

From this, concavity of entropy, (5.11) and $\theta^{\times k_j}(Z) > 1-\delta$, it follows that for $b \in B_j^{\omega}$,

$$\begin{aligned} &\frac{1}{m_j} H\Big(\theta^{*k_j}, \mathcal{E}^{\omega}_{b+\ell_j-a+m_j} \mid \mathcal{E}^{\omega}_{b+\ell_j-a} \vee \pi^{-1}_{[d]\setminus\{j\}} \mathcal{E}^{\omega}_{b+\ell_j-a+m_j}\Big) \geq \\ &\quad \int_Z \frac{1}{m_j} H\Big(*_{s=1}^{k_j} \theta^{\omega}_{x_s,b}, \mathcal{E}^{\omega}_{b+\ell_j-a+m_j} \mid \mathcal{C}^{\omega}_b\Big) \, \mathrm{d}\theta^{\times k_j}(x_1, \dots, x_{k_j}) \geq N\chi_j - O(\delta). \end{aligned} \tag{5.12}$$

Finally, define $Q^{\omega} := \Big\{q \in [n] \colon q - \ell_j + a \in B_j^{\omega}\Big\}$. From $\ell_j, a, \delta^{-1} \ll n$ and $\#_n(B_j^{\omega}) > \delta/(2d)$, it follows that $\#_n(Q^{\omega}) > \delta/(3d)$. The proof is finished by $\varepsilon^{-1} \ll \delta^{-1}$ and (5.12). □

6. Entropy of component measures

In this section, we prove three lemmas about the entropy of μ^{ω} across different scales, which will be applied in Sections 7 and 8. Lemma 6.1 is an analog of [46, Lemma 4.1], while Lemmas 6.2 and 6.3 replace [46, Lemmas 1.13 and 1.14] with analogous estimates for random measures at a large proportion of scales.

We begin with some notations. By Theorem 3.2, for $\mathbf{P}$-a.e. ω and $J \subset [d]$, $\pi_J \mu^{\omega}$ is exact dimensional with dimension given by $\dim \pi_J \mathcal{A}$ as in (3.4). Inspired by [46], we define

$$\kappa_{\mathcal{A}} := \sum_{j=1}^{d-1} \chi_j + \chi_d(\dim \mathcal{A} - (d-1)). \tag{6.1}$$

Now, we are ready to state the three lemmas to be proved in this section.

Lemma 6.1. *Suppose* $\dim \pi_{[d-1]} \mathcal{A} = d-1$. *For* $\eta \in (0,1)$ *there exists* $\overline{\Omega} \subset \Omega$ *with* $\mathbf{P}(\overline{\Omega}) > 1-\eta$ *such that for* $n \in \mathbb{N}$ *with* $\eta^{-1} \ll n$,

$$\left| \frac{1}{n} H(\mu^{\omega}, \mathcal{E}^{\omega}_n) - N\kappa_{\mathcal{A}} \right| < \eta \quad \textit{for } \omega \in \overline{\Omega}.$$

Lemma 6.2. *Suppose* $\dim \pi_{[d-1]} \mathcal{A} = d-1$. *For* $\eta \in (0,1)$ *there exists* $\overline{\Omega} \subset \Omega$ *with* $\mathbf{P}(\overline{\Omega}) \geq 1-\eta$ *so that the following holds. Let* $m, n \in \mathbb{N}$ *be with* $\eta^{-1} \ll m \ll n$. *Then for* $\omega \in \overline{\Omega}$ *there is* $Q^{\omega} \subset [n]$ *with* $\#_n(Q^{\omega}) \geq 1-\eta$ *so that*

$$\frac{1}{m} H\big(\mu^{\omega}, \mathcal{E}^{\omega}_{q+m} \mid \mathcal{E}^{\omega}_q\big) > N\kappa_{\mathcal{A}} - \eta \quad \textit{for } q \in Q^{\omega}.$$

Lemma 6.3. *Suppose* $\dim \pi_{[J]} \mathcal{A} = |J|$ *for some* $J \subset [d]$. *For* $\eta \in (0,1)$ *there exists* $\overline{\Omega} \subset \Omega$ *with* $\mathbf{P}(\overline{\Omega}) \geq 1-\eta$ *so that the following holds. Let* $m, n \in \mathbb{N}$ *be with* $\eta^{-1} \ll m \ll n$. *Then for* $\omega \in \overline{\Omega}$ *there is* $Q^{\omega} \subset [n]$ *with* $\#_n(Q^{\omega}) \geq 1-\eta$ *so that*

$$\frac{1}{m} H\big(\mu^{\omega}, \pi_J^{-1} \mathcal{E}^{\omega}_{q+m} \mid \mathcal{E}^{\omega}_q\big) > N \sum_{j \in J} \chi_j - \eta \quad \textit{for } q \in Q^{\omega}.$$

6.1. **Entropy growth along dyadic partitions.** In this subsection, we explore the entropy growth of the random measures along dyadic partitions.

Lemma 6.4. *For $\eta \in (0,1)$ there exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1-\eta$ so that for $n \in \mathbb{N}$ with $\eta^{-1} \ll n$ and $J \subset [d]$,*

$$\left| \frac{1}{n} H\Big(\pi_J \mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}}\Big) - N\chi_d \dim \pi_J \mathcal{A} \right| < \eta \quad \text{for } \omega \in \overline{\Omega}.$$

Proof. By Egorov's theorem, there is $\Omega_1 \subset \Omega$ with $\mathbf{P}(\Omega_1) > 1-\eta/2$ so that for $\omega \in \Omega_1$

$$\left| \frac{1}{n} H(\pi_J \mu^\omega, \mathcal{D}_{nN\chi_d}) - N\chi_d \dim \pi_J \mathcal{A} \right| < \eta/2.$$

On the other hand, by Lemma 4.1 there exists $\overline{\Omega} \subset \Omega_1$ with $\mathbf{P}(\overline{\Omega}) > 1-\eta$ so that for $\omega \in \overline{\Omega}$

$$\left| \chi_d^{\omega|n} - nN\chi_d \right| < n\eta/2.$$

The proof is finished by combining the above two equations with Lemma 2.2(iv). ☐

Lemma 6.5. *Suppose* $\dim \pi_J \mathcal{A} = |J|$ *for some* $J \subset [d]$. *For $\eta \in (0,1)$ there exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1-\eta$ so that for $n \in \mathbb{N}$ with $\eta^{-1} \ll n$,*

$$\left| \frac{1}{n} H\Big(\pi_J \mu^{T^n\omega}, \mathcal{D}_{\chi_d^{\omega|n} - \chi_1^{\omega|n}}\Big) - |J|N(\chi_d - \chi_1) \right| < \eta \quad \text{for } \omega \in \overline{\Omega}. \tag{6.2}$$

Proof. Let $\varepsilon \in (0,1)$ be with $\eta^{-1} \ll \varepsilon^{-1}$. By Egorov's theorem, there is $\Omega_1 \subset \Omega$ with $\mathbf{P}(\Omega_1) > 1-\varepsilon$ and $n_0 \in \mathbb{N}$ so that for $\omega \in \Omega_1$ and $n \geq n_0$,

$$\frac{1}{n} H(\pi_J \mu^\omega, \mathcal{D}_n) \geq |J| - \varepsilon.$$

Then by $\mathbf{P}$ being T-invariant, we have

$$\begin{aligned}\int \inf_{n \geq n_0} \frac{1}{n} H\big(\pi_{[d-1]} \mu^{T^n\omega}, \mathcal{D}_n\big) \, \mathrm{d}\mathbf{P}(\omega) &= \int \inf_{n \geq n_0} \frac{1}{n} H\big(\pi_{[d-1]} \mu^\omega, \mathcal{D}_n\big) \, \mathrm{d}\mathbf{P}(\omega) \\ &\geq \int_{\Omega_1} \inf_{n \geq n_0} \frac{1}{n} H\big(\pi_{[d-1]} \mu^\omega, \mathcal{D}_n\big) \, \mathrm{d}\mathbf{P}(\omega) \\ &\geq |J| - O(\varepsilon).\end{aligned}$$

On the other hand, we have $(1/n)H\big(\pi_J \mu^{T^n\omega}, \mathcal{D}_n\big) \leq |J|$ for $n \in \mathbb{N}$. From this and above, it follows that there exists $\Omega_2 \subset \Omega$ with $\mathbf{P}(\Omega_2) > 1 - O\big(\varepsilon^{1/3}\big)$ so that for $\omega \in \Omega_2$ and $n \geq n_0$,

$$\left| \frac{1}{n} H\big(\pi_J \mu^{T^n\omega}, \mathcal{D}_n\big) - |J| \right| \leq O\Big(\varepsilon^{1/3}\Big). \tag{6.3}$$

By Lemma 4.1 there is $\overline{\Omega} \subset \Omega_2$ with $\mathbf{P}(\overline{\Omega}) > 1 - O\big(\varepsilon^{1/3}\big)$ so that for $\omega \in \overline{\Omega}$ and $\varepsilon^{-1} \ll n$,

$$\left| \chi_d^{\omega|n} - \chi_1^{\omega|n} - nN(\chi_d - \chi_1) \right| \leq n\varepsilon. \tag{6.4}$$

Combining (6.3) and (6.4), we conclude from Lemma 2.2(iv) that for $\omega \in \overline{\Omega}$ and $\varepsilon^{-1} \ll n$,

$$\left| \frac{1}{n} H\Big(\pi_J \mu^{T^n\omega}, \mathcal{D}_{\chi_d^{\omega|n} - \chi_1^{\omega|n}}\Big) - |J|N(\chi_d - \chi_1) \right| < O\Big(\varepsilon^{1/3}\Big).$$

This finishes the proof since $\eta^{-1} \ll \varepsilon^{-1}$. ☐

6.2. **Proof of Lemmas 6.1–6.3.** In this subsection, we prove the lemmas in the beginning of this section. First we prove Lemma 6.1.

Proof of Lemma 6.1. Let $\varepsilon \in (0,1)$ be with $\eta^{-1} \ll \varepsilon^{-1} \ll n$. Let $\overline{\Omega}$ be the intersection of the $\overline{\Omega}$'s obtained by applying Lemmas 4.1, 6.4 and 6.5 with ε, n in place of η, n. Then $\mathbf{P}(\overline{\Omega}) > 1-3\varepsilon$. In what follows we take $\omega \in \overline{\Omega}$. Note that by (4.6),

$$H(\mu^\omega, \mathcal{E}_n^\omega) = H\left(\mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}}\right) - H\left(\mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^\omega\right) + O(1)\,.$$

From this, Lemma 6.4, (6.1) and $\eta^{-1} \ll \varepsilon^{-1}$, it suffices to show

$$\frac{1}{n} H\left(\mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^\omega\right) = N \sum_{j=1}^{d-1} (\chi_d - \chi_j) + O(\varepsilon)\,. \tag{6.5}$$

First we show the upper bound. It follows from Lemma 4.1 that for each $E \in \mathcal{E}_n^\omega$,

$$\log \#\left\{D \in \mathcal{D}_{\chi_d^{\omega|n}} : D \cap E \neq \emptyset\right\} \le \sum_{j=1}^{d} (\chi_d^{\omega|n} - \chi_j^{\omega|n}) + O(1) \le nN \sum_{j=1}^{d} (\chi_d - \chi_j) + O(n\varepsilon)\,.$$

Thus,

$$\frac{1}{n} H\left(\mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^\omega\right) \le N \sum_{j=1}^{d-1} (\chi_d - \chi_j) + O(\varepsilon)\,. \tag{6.6}$$

Next, we prove the lower bound in (6.5). Since $A^{-\omega|n}\mathcal{D}_{\chi_d^{\omega|n}}$ and $\pi_{[d-1]}^{-1}\left(\bigtimes_{j=1}^{d-1} \mathcal{D}_{\chi_d^{\omega|n}-\chi_j^{\omega|n}}\right)$ are $O(1)$-commensurable, it follows from (4.3), Lemma 2.2(iii) and (4.7) that

$$\begin{aligned}
H\left(\mu^\omega, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^\omega\right) &= H\left(\nu_n^\omega * A^{\omega|n}\mu^{T^n\omega}, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^\omega\right) \\
&\ge H\left(\mu^{T^n\omega}, (A^{\omega|n})^{-1}\mathcal{D}_{\chi_d^{\omega|n}}\right) - O(1) \\
&\ge H\left(\mu^{T^n\omega}, \pi_{[d-1]}^{-1}\left(\bigtimes_{j=1}^{d-1} \mathcal{D}_{\chi_d^{\omega|n}-\chi_j^{\omega|n}}\right)\right) - O(1) \\
&= H\left(\pi_{[d-1]}\mu^{T^n\omega}, \bigtimes_{j=1}^{d-1} \mathcal{D}_{\chi_d^{\omega|n}-\chi_j^{\omega|n}}\right) - O(1)\,.
\end{aligned} \tag{6.7}$$

For each $E \in \bigtimes_{j=1}^{d-1} \mathcal{D}_{\chi_d^{\omega|n}-\chi_j^{\omega|n}}$, by Lemma 4.1 we have

$$\log \#\left\{F \in \mathcal{D}^{d-1}_{\chi_d^{\omega|n}-\chi_1^{\omega|n}} : F \cap E \neq \emptyset\right\} \le \sum_{j=1}^{d-1} (\chi_j^{\omega|n} - \chi_1^{\omega|n}) + O(1) \le nN \sum_{j=1}^{d-1} (\chi_j - \chi_1) + O(n\varepsilon)\,.$$

Thus,

$$\frac{1}{n} H\left(\pi_{[d-1]}\mu^{T^n\omega}, \mathcal{D}^{d-1}_{\chi_d^{\omega|n}-\chi_1^{\omega|n}} \mid \bigtimes_{j=1}^{d-1} \mathcal{D}_{\chi_d^{\omega|n}-\chi_j^{\omega|n}}\right) \le N \sum_{j=1}^{d-1} (\chi_j - \chi_1) + O(\varepsilon)\,. \tag{6.8}$$

Applying Lemma 2.1(v), we conclude from (6.7), (6.8) and Lemma 6.5 that

$$\frac{1}{n}H\Big(\mu^{\omega}, \mathcal{D}_{\chi_d^{\omega|n}} \mid \mathcal{E}_n^{\omega}\Big) \geq (d-1)N(\chi_d - \chi_1) - N\sum_{j=1}^{d-1}(\chi_j - \chi_1) - O(\varepsilon)$$
$$= N\sum_{j=1}^{d-1}(\chi_d - \chi_j) - O(\varepsilon)\,.$$

This, together with (6.6), finishes the proof of (6.5). □

Next, we prove Lemma 6.2.

Proof of Lemma 6.2. By applying Lemma 6.1 with $\eta/2, m$ in place of η, n, there exists $\Omega_1 \subset \Omega$ with $\mathbf{P}(\Omega_1) > 1 - \eta/2$ so that for $\omega \in \Omega_1$,

$$\frac{1}{m}H(\mu^{\omega}, \mathcal{E}_m^{\omega}) > N\kappa_{\mathcal{A}} - \frac{\eta}{2}.$$

By applying Birkhoff's ergodic theorem and Egorov's theorem to $\mathbf{1}_{\Omega_1}$, we find $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1 - \eta$ such that for $\omega \in \overline{\Omega}$ there is $Q^{\omega} \subset [n]$ with $\#_n(Q^{\omega}) > 1 - \eta$ and $T^q\omega \in \Omega_1$ for $q \in Q^{\omega}$. From the above inequality, $T^q\omega \in \Omega_1$, $\eta^{-1} \ll m$ and (4.9), it follows that

$$\frac{1}{m}H\big(\mu^{\omega}, \mathcal{E}_{q+m}^{\omega} \mid \mathcal{E}_q^{\omega}\big) \geq \frac{1}{m}H\big(\mu^{T^q\omega}, \mathcal{E}_m^{T^q\omega}\big) - O\Big(\frac{1}{m}\Big) > N\kappa_{\mathcal{A}} - \eta.$$

This finishes the proof. □

Finally, we prove Lemma 6.3.

Proof of Lemma 6.3. Let $\varepsilon \in (0,1)$ be with $\eta^{-1} \ll \varepsilon^{-1} \ll m$. Let Ω_1 be the intersection of the $\overline{\Omega}$'s obtained from applying Lemma 4.1 and Lemma 6.4 with ε, m in place of η, n. Then $\mathbf{P}(\Omega_1) > 1 - 2\varepsilon$. By applying Birkhoff's ergodic theorem and Egorov's theorem to $\mathbf{1}_{\Omega_1}$, we find $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1 - \eta$ so that for $\omega \in \overline{\Omega}$ there is $Q^{\omega} \subset [n]$ with $\#_n(Q^{\omega}) > 1 - \eta$ and $T^q\omega \in \Omega_1$ for $q \in Q^{\omega}$. In what follows we take $\omega \in \Omega$ and let $q \in Q^{\omega}$. Then $T^q\omega \in \Omega_1$.

For $E \in \mathcal{E}_m^{T^q\omega}$ with $E \cap \pi_J\mathbb{R}^d \neq \emptyset$, by Lemma 4.1 we have

$$\log\#\Big\{D \in \mathcal{D}_{\chi_d^{T^q\omega|m}} : D \cap E \cap \pi_J\mathbb{R}^d \neq \emptyset\Big\} \leq mN\sum_{j\in J}(\chi_d - \chi_j) + O(m\varepsilon)\,.$$

Thus,

$$\frac{1}{m}H\Big(\pi_J\mu^{T^q\omega}, \mathcal{D}_{\chi_d^{T^q\omega|m}} \mid \mathcal{E}_m^{T^q\omega}\Big) \leq N\sum_{j\in J}(\chi_d - \chi_j) + O(\varepsilon)\,. \tag{6.9}$$

Next we estimate that

$$\begin{aligned}
&H\big(\mu^{\omega}, \pi_J^{-1}\mathcal{E}_{q+m}^{\omega} \mid \mathcal{E}_q^{\omega}\big)\\
&\geq H\Big(A^{\omega|q}\mu^{T^q\omega}, \pi_J^{-1}\mathcal{E}_{q+m}^{\omega} \mid \mathcal{E}_q^{\omega}\Big) && \text{(by (4.3) and concavity of entropy)}\\
&\geq H\big(\mu^{T^q\omega}, \pi_J^{-1}\mathcal{E}_m^{T^q\omega}\big) - O(1) && \text{(by (4.5))}\\
&= H\big(\pi_J\mu^{T^q\omega}, \mathcal{E}_m^{T^q\omega}\big) - O(1) && \text{(by Lemma 2.1(iii))}\\
&= H\Big(\pi_J\mu^{T^q\omega}, \mathcal{D}_{\chi_d^{T^q\omega|m}}\Big) - H\Big(\pi_J\mu^{T^q\omega}, \mathcal{D}_{\chi_d^{T^q\omega|m}} \mid \mathcal{E}_m^{T^q\omega}\Big) - O(1)\,,
\end{aligned}$$

where the last equality is by Lemma 2.1(v). Since $T^q\omega \in \Omega_1$ and $\eta^{-1} \ll m$, combining the above with Lemma 6.4 and (6.9) yields that

$$\frac{1}{m}H\left(\mu^\omega, \pi_J^{-1}\mathcal{E}^\omega_{q+m} \mid \mathcal{E}^\omega_q\right) \geq N\sum_{j\in J}\chi_j - O\left(\frac{1}{m}\right) - O(\varepsilon)\,.$$

This finishes the proof since $\eta^{-1} \ll \varepsilon^{-1} \ll m$. □

7. Proof of the entropy increase result

In this section, we prove the following entropy increase result for random measures, which serves as an analog to [46, Theorem 1.12]. This result is a crucial ingredient in the proof of Theorem 1.12.

Theorem 7.1. *Suppose* $\dim \mathcal{A} < d$ *and* $\dim \pi_J\mathcal{A} = |J|$ *for each* $J \subsetneq [d]$*. For* $\varepsilon \in (0,1)$ *there exists* $\delta = \delta(\varepsilon) > 0$ *so that the following holds. Let* $\eta \in (0,1)$ *be with* $\varepsilon^{-1} \ll \eta^{-1}$*. There exists* $\overline{\Omega} \subset \Omega$ *with* $\mathbf{P}(\overline{\Omega}) > 1-\eta$ *so that for* $n \in \mathbb{N}$ *with* $\eta^{-1} \ll n$ *and* $\omega \in \overline{\Omega}$ *the following holds. Let* $\theta \in \mathcal{M}_c(\mathbb{R}^d)$ *with* $\operatorname{diam}(\operatorname{supp}\theta) \leq 1/\varepsilon$ *and* $\frac{1}{n}H(\theta, \mathcal{E}^\omega_n) > \varepsilon$*. Then* $\frac{1}{n}H(\theta * \mu^\omega, \mathcal{E}^\omega_n) \geq N\kappa_{\mathcal{A}} + \delta$*.*

To prove Theorem 7.1, we need the following version of the Kaimanovich-Vershik lemma. Its proof follows a similar approach of [46, Corollary 5.2] and is therefore omitted.

Lemma 7.2. *Let* $\omega \in \Omega$*,* $\theta, \rho \in \mathcal{M}_c(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$ *be given. Then for* $k \in \mathbb{N}$*,*

$$H\left(\theta^{*k} * \rho, \mathcal{E}^\omega_n\right) - H(\rho, \mathcal{E}^\omega_n) \leq k\left(H(\theta * \rho, \mathcal{E}^\omega_n) - H(\rho, \mathcal{E}^\omega_n)\right) + O(k)\,.$$

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. The proof is adapted from [46, Theorem 1.12], with Proposition 5.1, Lemmas 6.2 and 6.3 respectively in place of [46, Proposition 1.15, Lemmas 1.13 and 1.14]. To account for the dependence on additional parameters and for clarity, we provide the necessary details.

Let $\delta_0, \varepsilon_1 \in (0,1)$ and $m_1, \ldots, m_d, k_1, \ldots, k_d \in \mathbb{N}$ be with

$$\varepsilon^{-1} \ll \delta_0^{-1} \ll \eta^{-1} \ll m_d \ll k_d \ll \cdots \ll m_1 \ll k_1 \ll \varepsilon_1^{-1} \ll n. \tag{7.1}$$

Let $\overline{\Omega}$ be the intersection of the $\overline{\Omega}$'s obtained by applying Proposition 5.1 with $\varepsilon, \delta_0, \eta, m_j, k_j$ in place of $\varepsilon, \delta, \eta, m_j, k_j$, Lemmas 6.2 with η in place of η, Lemma 6.3 repeatedly for $J \subsetneq [d]$ with J, η in place of J, η, and Lemma 6.1 with ε_1 in place of η. Then $\mathbf{P}(\overline{\Omega}) > 1 - O(\eta)$. Note that $\eta^{-1} \ll m_j \ll k_j \ll n$ for $1 \leq j \leq d$. For $\omega \in \overline{\Omega}$, let $Q^\omega_1, Q^\omega_2, Q^\omega_3$ be respectively the Q^ω obtained from Proposition 5.1, Lemmas 6.2 and 6.3. Then $\#_n(Q^\omega_1) > \delta_0$, $\#_n(Q^\omega_2) > 1 - \eta/4$ and $\#_n(Q^\omega_3) > 1 - \eta/4$. Define $Q^\omega := Q^\omega_1 \cap Q^\omega_2 \cap Q^\omega_3$. From $\delta_0^{-1} \ll \eta^{-1}$, it follows that $\#_n(Q^\omega) > \delta_0 - \eta/2 > \delta_0/2$. Let $1 \leq j \leq d$ be the integer obtained along with Q^ω_1 in the application of Proposition 5.1. In what follows we take $\omega \in \overline{\Omega}$, and let $Q^\omega \subset [n]$ accordingly.

Note that $\operatorname{diam}(\operatorname{supp}\theta^{*k_j}) \le k_j/\varepsilon$ and $\varepsilon^{-1} \ll \eta^{-1} \ll m_j \ll k_j \ll n$. Using Lemma 4.2, it follows that

$$
\begin{aligned}
&\frac{1}{n}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_n^\omega\Big) \\
&\ge \mathbb{E}_{1\le q\le n}\left(\frac{1}{m_j}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big)\right) - O(\eta) \\
&\ge \#_n(Q^\omega)\mathbb{E}_{q\in Q^\omega}\left(\frac{1}{m_j}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big)\right) \\
&\quad + \#_n(Q_2^\omega \setminus Q^\omega)\mathbb{E}_{q\in Q_2^\omega\setminus Q^\omega}\left(\frac{1}{m_j}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big)\right) - O(\eta)\,.
\end{aligned}
\tag{7.2}
$$

By $\dim\mathcal{A} < d$, we have $\Delta := \sum_{j=1}^d \chi_j - \kappa_{\mathcal{A}} > 0$. By Lemma 2.1(v), concavity of entropy, (4.7) and $\eta^{-1} \ll m_j$, we conclude from Proposition 5.1 and Lemma 6.3 that for $q \in Q^\omega$,

$$
\begin{aligned}
\frac{1}{m_j}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big) &\ge \frac{1}{m_j}H\Big(\theta^{*k_j}, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega \vee \pi_{[d]\setminus\{j\}}^{-1}\mathcal{E}_{q+m_j}^\omega\Big) \\
&\quad + \frac{1}{m_j}H\Big(\mu^\omega, \pi_{[d]\setminus\{j\}}^{-1}\mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big) - O\Big(\frac{1}{m_j}\Big) \\
&\ge N\chi_j + N\sum_{j'\ne j}\chi_{j'} - O(\eta) \\
&= N\kappa_{\mathcal{A}} + N\Delta - O(\eta)\,.
\end{aligned}
\tag{7.3}
$$

For $q \in Q_2^\omega$, by concavity of entropy and $\eta^{-1} \ll m_j$, it follows from Lemma 6.2 that

$$
\frac{1}{m_j}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big) \ge \frac{1}{m_j}H\Big(\mu^\omega, \mathcal{E}_{q+m_j}^\omega \mid \mathcal{E}_q^\omega\Big) - O\Big(\frac{1}{m_j}\Big) > N\kappa_{\mathcal{A}} - O(\eta)\,.
\tag{7.4}
$$

Combining (7.2), (7.3), (7.4), $\#_n(Q^\omega) > \delta_0/2$ and $\#_n(Q_2^\omega) > 1-\eta/4$ shows that

$$
\begin{aligned}
\frac{1}{n}H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_n^\omega\Big) &\ge N\kappa_{\mathcal{A}} + \frac{\delta_0 N\Delta}{2} - O(\eta) \\
&\ge \frac{1}{n}H(\mu^\omega, \mathcal{E}_n^\omega) + \frac{\delta_0 N\Delta}{2} - O(\eta) && \text{(by Lemma 6.1)} \\
&\ge \frac{1}{n}H(\mu^\omega, \mathcal{E}_n^\omega) + \delta_0^2. && \text{(by } \delta_0^{-1} \ll \eta^{-1}\text{)}
\end{aligned}
$$

By a rearrangement,

$$
\frac{1}{n}\left(H\Big(\theta^{*k_j} * \mu^\omega, \mathcal{E}_n^\omega\Big) - H(\mu^\omega, \mathcal{E}_n^\omega)\right) \ge \delta_0^2.
$$

By Lemma 7.2 and $\delta_0^{-1} \ll k_j \ll n$,

$$
\frac{1}{n}\left(H(\theta * \mu^\omega, \mathcal{E}_n^\omega) - H(\mu^\omega, \mathcal{E}_n^\omega)\right) \ge \frac{\delta_0^2}{2k_j}.
$$

By Lemma 6.1 and $\delta_0^{-1} \ll k_j \ll \varepsilon_1^{-1}$, this completes the proof with $\delta = \delta_0^2/4k_j$. ☐

8. Proof of Theorem 1.12

In this section, we establish the following theorem, which directly implies Theorem 1.12.

For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the partition of $\Lambda^{\mathbb{N}}$ defined by that $\mathcal{C}_n(x) = \mathcal{C}_n(y)$ if and only if $\varphi_{x|n} = \varphi_{y|n}$ for $x, y \in \Lambda^{\mathbb{N}}$.

Theorem 8.1. *Fix $N \in \mathbb{N}$. Let Γ be a partition of $\Lambda^{\mathbb{N}}$ satisfying (4.1). Set $\mathcal{A} = \vee_{i=0}^{\infty}\sigma^{-iN}\Gamma$. Suppose $\chi_1 < \cdots < \chi_d$, and Φ_j is Diophantine for $1 \le j \le d$. Suppose further that for $x, y \in \Lambda^{\mathbb{N}}, n \in \mathbb{N}$ and $1 \le j \le d$, $\pi_j\varphi_{x|n} = \pi_j\varphi_{y|n}$ implies $\varphi_{x|n} = \varphi_{y|n}$. Then*

$$\dim \mathcal{A} = \min\{d, f_\Phi(h_{RW}(\Phi, \mathcal{A}))\}, \tag{8.1}$$

where $\dim \mathcal{A}$ is from Theorem 3.2, f_Φ is as in (1.5), and $h_{RW}(\Phi, \mathcal{A})$ is as in (1.18).

8.1. **Super-exponential concentration.** Using Theorem 7.1, we derive the following theorem, which demonstrates that any linear acceleration of scales fails to produce positive entropy for ν_n^ω. This indicates a super-exponential concentration of cylinders.

Theorem 8.2. *If $\dim \mathcal{A} < d$ and $\dim \pi_{[J]}\mathcal{A} = |J|$ for each $J \subsetneq [d]$. Then for $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$ with $\varepsilon^{-1} \ll n$, there exists $\overline{\Omega} \subset \Omega$ with $\mathbf{P}(\overline{\Omega}) > 1 - \varepsilon$ so that*

$$\frac{1}{n}H(\nu_n^\omega, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega) < \varepsilon \quad \text{for } \omega \in \overline{\Omega}. \tag{8.2}$$

Proof. Suppose on the contrary that there exist $M > 1$, $\varepsilon \in (0,1)$, $n \in \mathbb{N}$ with $\varepsilon^{-1} \ll n$, and $\Omega_1 \subset \Omega$ with $\mathbf{P}(\Omega_1) \ge \varepsilon$ so that for $\omega \in \Omega_1$,

$$\frac{1}{n}H(\nu_n^\omega, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega) \ge \varepsilon. \tag{8.3}$$

Let $\eta \in (0,1)$ be with

$$\varepsilon^{-1}, M \ll \eta^{-1} \ll n. \tag{8.4}$$

Let Ω_2 be the intersection of the $\overline{\Omega}$'s obtained from Lemma 6.1 and Theorem 7.1 with ε, η in place of ε, η. Then $\mathbf{P}(\Omega_2) > 1 - 2\eta$. Define $\Omega_3 := \Omega_1 \cap \Omega_2 \cap T^{-n}\Omega_2$. Since $\mathbf{P}$ is T-invariant, $\mathbf{P}(T^{-n}\Omega_2) = \mathbf{P}(\Omega_2) > 1 - 2\eta$. By $\varepsilon^{-1} \ll \eta^{-1}$ we have $\mathbf{P}(\Omega_3) > \varepsilon - 4\eta > \varepsilon/2 > 0$. In what follows we take $\omega \in \Omega_3$.

For $x \in \mathbb{R}^d$, define $\theta_x^\omega := A^{-\omega|n}(\nu_n^\omega)_{\mathcal{E}_n^\omega(x)}$. Then $\operatorname{diam}(\operatorname{supp}\theta_x^\omega) = O(1)$. Combining (4.5), (4.11) and (8.3) yields that

$$\int \frac{1}{n}H\Big(\theta_x^\omega, \mathcal{E}_{(M-1)n}^{T^n\omega}\Big)\, \mathrm{d}\nu_n^\omega(x) = \int \frac{1}{n}H\big((\nu_n^\omega)_{\mathcal{E}_n^\omega(x)}, \mathcal{E}_{Mn}^\omega\big)\, \mathrm{d}\nu_n^\omega(x) = \frac{1}{n}H(\nu_n^\omega, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega) \ge \varepsilon.$$

Since $\frac{1}{n}H\Big(\theta_x^\omega, \mathcal{E}_{(M-1)n}^{T^n\omega}\Big) \le C(M-1)$ for some $C > 0$, from above there exists $E \subset \mathbb{R}^d$ with $\nu_n^\omega(E) > \varepsilon/(4C(M-1))$ so that for $x \in E$,

$$\frac{1}{n}H\Big(\theta_x^\omega, \mathcal{E}_{(M-1)n}^{T^n\omega}\Big) > \frac{\varepsilon}{4}.$$

Hence by $T^n\omega \in \Omega_2$ and Theorem 7.1 there exists $\delta = \delta(\varepsilon, M) > 0$ so that

$$\frac{1}{n}H\Big(\theta_x^\omega * \mu^{T^n\omega}, \mathcal{E}_{(M-1)n}^{T^n\omega}\Big) \ge (M-1)N\kappa_{\mathcal{A}} + (M-1)\delta. \tag{8.5}$$

By $\omega, T^n\omega \in \Omega_2$ and $M \ll \eta^{-1} \ll n$, it follows from Lemma 6.1 that

$$\frac{1}{n}H\Big(\mu^{T^n\omega}, \mathcal{E}_{(M-1)n}^{T^n\omega}\Big) > (M-1)N\kappa_{\mathcal{A}} - O(\eta), \tag{8.6}$$

and

$$\frac{1}{n}H(\mu, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega) < (M-1)N\kappa_{\mathcal{A}} + O(\eta). \tag{8.7}$$

Note that $\operatorname{diam}\left(\operatorname{supp}\theta_x^\omega * \mu^{T^n\omega}\right) = O(1)$. From all above we estimate that,

$$\begin{aligned}
&(M-1)N\kappa_{\mathcal{A}} + O(\eta) \\
&\quad \geq \frac{1}{n}H(\mu, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega) && \text{(by (8.7))} \\
&\quad = \frac{1}{n}H\left(\nu_n^\omega * A^{\omega|n}\mu^{T^n\omega}, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega\right) && \text{(by (4.3))} \\
&\quad \geq \int \frac{1}{n}H\left((\nu_n^\omega)_{\mathcal{E}_n^\omega(x)} * A^{\omega|n}\mu^{T^n\omega}, \mathcal{E}_{Mn}^\omega \mid \mathcal{E}_n^\omega\right) \,\mathrm{d}\nu_n^\omega(x) && \text{(by concavity of entropy)} \\
&\quad \geq \int \frac{1}{n}H\left(\theta_x^\omega * \mu^{T^n\omega}, \mathcal{E}_{(M-1)n}^{T^n\omega}\right) \,\mathrm{d}\nu_n^\omega(x) - O(\eta) && \text{(by (4.5))} \\
&\quad \geq \int_{\mathbb{R}^d\setminus E} \frac{1}{n}H\left(\mu^{T^n\omega}, \mathcal{E}_{(M-1)n}^{T^n\omega}\right) \,\mathrm{d}\nu_n^\omega(x) && \text{(by concavity of entropy and (4.7))} \\
&\qquad + \int_E \frac{1}{n}H\left(\theta_x^\omega * \mu^{T^n\omega}, \mathcal{E}_{(M-1)n}^{T^n\omega}\right) \,\mathrm{d}\nu_n^\omega(x) - O(\eta) \\
&\quad \geq (1-\nu_n^\omega(E))((M-1)N\kappa_{\mathcal{A}} - O(\eta)) && \text{(by (8.6))} \\
&\qquad + \nu_n^\omega(E)((M-1)N\kappa_{\mathcal{A}} + (M-1)\delta) - O(\eta) && \text{(by (8.5))} \\
&\quad = (M-1)N\kappa_{\mathcal{A}} + \frac{\varepsilon\delta}{4C} - O(\eta)\,. && \text{(by } \nu_n^\omega(E) > \varepsilon/(4C(M-1)))
\end{aligned}$$

Then a rearrangement shows that

$$\frac{\varepsilon\delta}{C} < O(\eta)\,.$$

This contradicts $\delta = \delta(\varepsilon, M)$ and $\varepsilon^{-1}, M \ll \eta^{-1}$. The proof is completed. ☐

8.2. **Proof of Theorem 8.1.** We begin with a lemma that relates the entropies of ν_n^ω and μ^ω.

Lemma 8.3. *Let $\eta \in (0,1)$ and $n \in \mathbb{N}$ be with $\eta^{-1} \ll n$. Then for $\omega \in \Omega$,*

$$\left|\frac{1}{n}H(\nu_n^\omega, \mathcal{E}_n^\omega) - \frac{1}{n}H(\mu^\omega, \mathcal{E}_n^\omega)\right| < \eta.$$

Proof. Define $\Pi^{nN} : \Lambda^{\mathbb{N}} \to \mathbb{R}^d$ by $\Pi^{nN}(x) = \varphi_{x|nN}(0)$ for $x \in \Lambda^{\mathbb{N}}$. Since $\mu^\omega = \Pi\beta^\omega$, $\nu_n^\omega = \Pi^{nN}\beta^\omega$, and $\left|\pi_j\left(\Pi(x) - \Pi^{nN}(x)\right)\right| \leq O\left(\lambda_j^{\omega|n}\right)$ for $1 \leq j \leq d$, the proof is finished by (4.8). ☐

Next, we give some properties of the function defined in (1.5). Let $1 \leq j_1 < \cdots < j_s \leq d$ and write $J = \{j_b\}_{b=1}^s$. Recall the IFS Φ_J from (2.1). By (1.5),

$$f_{\Phi_J}(x) = \begin{cases} \ell + \frac{x - \sum_{b=1}^{\ell}\chi_{j_b}}{\chi_{j_{\ell+1}}} & \text{if } x \in \left[\sum_{b=1}^{\ell}\chi_{j_b}, \sum_{b=1}^{\ell+1}\chi_{j_b}\right) \text{ for some } 0 \leq \ell \leq s-1; \\ s\frac{x}{\sum_{b=1}^s \chi_{j_b}} & \text{if } x \in [\sum_{b=1}^s \chi_{j_b}, \infty)\,. \end{cases} \tag{8.8}$$

The following two lemmas provide the desired properties of f_{Φ_J}. Their proofs follow directly from the definition and are thus omitted.

Lemma 8.4. *For $x \geq 0$, write*

$$Y(x) := \left\{(y_1, \ldots, y_s) \in \mathbb{R}^s : 0 \leq y_b \leq \chi_{j_b} \text{ for } 1 \leq b \leq s \text{ and } \sum_{b=1}^s y_b \leq x\right\},$$

and let $g\colon Y(x)\to[0,\infty)$ *be defined as*

$$g(y)=\sum_{b=1}^{s}\frac{y_b}{\chi_{j_b}}\quad \textit{for } y=(y_1,\ldots,y_s)\in Y(x).$$

If $f_{\Phi_J}(x)\le s$*, then* $\max_{y\in Y(x)} g(y)=f_{\Phi_J}(x)$ *and the maximal value is uniquely attained at*

$$\widetilde{y}:=\left(\chi_{j_1},\ldots,\chi_{j_m},x-\sum_{b=1}^{m}\chi_{j_b},0,\ldots,0\right),$$

where $m=\max\{0\le k\le s\colon \sum_{b=1}^{k}\chi_{j_b}\le x\}$.

Lemma 8.5. *For* $x\ge 0$ *and* $0\le m<s$*,*

$$m+\frac{x-\sum_{b=1}^{m}\chi_{j_b}}{\chi_{j_{m+1}}}\ge\min\left\{s,f_{\Phi_J}(x)\right\}.$$

Now we are ready to prove Theorem 8.1.

Proof of Theorem 8.1. The proof is adapted from [46, Theorem 1.7] and proceeds by induction on d. To address the parameter dependence arising from disintegration and to maintain clarity, we include all necessary details. Assume that the theorem holds whenever the dimension of the ambient space is strictly less than d. For $d=1$, this induction hypothesis is vacuous.

Let $\emptyset\ne J\subsetneq[d]$. Since $\pi_J\varphi_{x|n}=\pi_J\varphi_{y|n}$ implies $\varphi_{x|n}=\varphi_{y|n}$, the partitions $(\mathcal{C}_n)_{n\in\mathbb{N}}$ are the same for Φ_J and Φ. Thus $h_{RW}(\Phi_J,\mathcal{A})=h_{RW}(\Phi,\mathcal{A})$ by (1.18). Since $A_{\varphi_{x|n}}=A_{\varphi_{y|n}}$ implies $A_{\pi_J\varphi_{x|n}}=A_{\pi_J\varphi_{y|n}}$, the partition $\mathcal{A}$ also satisfies the assumption in the theorem for Φ_J. Note that $\dim\pi_J\mathcal{A}$ is the dimension of $\pi_J\Pi\beta^\omega=\Pi^{\Phi_J}\beta^\omega$ for $\mathbf{P}$-a.e. ω, where Π^{Φ_J} is the coding map associated with Φ_J. Hence by the induction hypothesis,

$$\dim\pi_J\mathcal{A}=\min\left\{|J|,f_{\Phi_J}(h_{RW}(\Phi,\mathcal{A}))\right\}\quad\text{for } \emptyset\ne J\subsetneq[d]. \tag{8.9}$$

Since combining Theorem 3.2 and Lemma 8.4 implies that $f_\Phi(h_{RW}(\Phi,\mathcal{A}))$ is always an upper bound of $\dim\mathcal{A}$, we only need to show that if $\dim\mathcal{A}<d$, then

$$\dim\mathcal{A}\ge\min\left\{d,f_\Phi(h_{RW}(\Phi,\mathcal{A}))\right\}.$$

In what follows we assume $\dim\mathcal{A}<d$.

First, suppose that $\dim\pi_{[d-1]}\mathcal{A}<d-1$. Then $\dim\pi_{[d-1]}\mathcal{A}=f_{\Phi_{[d-1]}}(h_{RW}(\Phi,\mathcal{A}))$ by (8.9). It follows from (8.8) that $f_{\Phi_{[d-1]}}(h_{RW}(\Phi,\mathcal{A}))=f_\Phi(h_{RW}(\Phi,\mathcal{A}))$. Hence $\dim\mathcal{A}\ge\dim\pi_{[d-1]}\mathcal{A}=f_\Phi(h_{RW}(\Phi,\mathcal{A}))$.

Next, suppose $\dim\pi_{[d-1]}\mathcal{A}=d-1$ and $\dim\pi_J\mathcal{A}<|J|$ for some $\emptyset\ne J\subsetneq[d]$. Then $\dim\pi_J\mathcal{A}=f_{\Phi_J}(h_{RW}(\Phi,\mathcal{A}))$ by (8.9). Write $J=\{j_b\}_{b=1}^{s}$ with $j_1<\cdots<j_s$, and set $J_b=\{j_1,\ldots,j_b\}$ for $0\le b\le s$. It follows from Theorem 3.2 that

$$\sum_{b=1}^{s}\frac{\Delta_b}{\chi_{j_b}}=f_{\Phi_J}(h_{RW}(\Phi,\mathcal{A})),$$

where $\Delta_b:=\mathrm{h}_{J_{b-1}}^{\mathcal{C},\mathcal{A}}-\mathrm{h}_{J_b}^{\mathcal{C},\mathcal{A}}\le\chi_{j_b}$ for $1\le b\le s$. Recall $\mathrm{h}_{\emptyset}^{\mathcal{C},\mathcal{A}}=h_{RW}(\Phi,\mathcal{A})$ by definition. Then Lemma 8.4 implies that $h_{RW}(\Phi,\mathcal{A})-\mathrm{h}_J^{\mathcal{C},\mathcal{A}}=\sum_{b=1}^{s}\Delta_b=h_{RW}(\Phi,\mathcal{A})$, and so $\mathrm{h}_J^{\mathcal{C},\mathcal{A}}=0$. This shows $\mathrm{h}^{\mathcal{C},\mathcal{A}}_{[d]} = 0$ by (3.5) and $\xi_J \prec \xi_{[d]}$. From $\dim \pi_{[d-1]}\mathcal{A} = d-1$ and Lemma 8.4 it follows that

$$\mathrm{h}^{\mathcal{C},\mathcal{A}}_{[j-1]} - \mathrm{h}^{\mathcal{C},\mathcal{A}}_{[j]} = \chi_j \quad \text{for } 1 \le j \le d-1.$$

Thus,

$$\mathrm{h}^{\mathcal{C},\mathcal{A}}_{[d-1]} - \mathrm{h}^{\mathcal{C},\mathcal{A}}_{[d]} = h_{RW}(\Phi, \mathcal{A}) - \sum_{j=1}^{d-1} \chi_j.$$

Combining the last two equations with Theorem 3.2 gives

$$\dim \mathcal{A} = \sum_{j=1}^{d} \frac{\mathrm{h}^{\mathcal{C},\mathcal{A}}_{[j-1]} - \mathrm{h}^{\mathcal{C},\mathcal{A}}_{[j]}}{\chi_j} = d - 1 + \frac{h_{RW}(\Phi, \mathcal{A}) - \sum_{j=1}^{d-1} \chi_j}{\chi_d} \ge \min\{d, f_\Phi(h_{RW}(\Phi, \mathcal{A}))\},$$

where the last inequality is by Lemma 8.5.

Finally, suppose $\dim \pi_{[d-1]}\mathcal{A} = d-1$ and $\dim \pi_J \mathcal{A} = |J|$ for each $J \subsetneq [d]$. Recall $S_n(\Phi_j)$, $1 \le j \le d$ from (1.7). For $n \in \mathbb{N}$, define $S_n(\Phi) = \max_{1\le j\le d} S_n(\Phi_j)$, and for $\omega \in \Omega$, define

$$S^\omega_n(\Phi) = \min\left\{ \max_{1\le j\le d} d(\varphi_{u,j}, \varphi_{v,j}) \colon u, v \in \Lambda^{nN}, \beta^\omega([u]) > 0, \beta^\omega([v]) > 0, \psi_u \ne \psi_v \right\},$$

with convention $\min \emptyset = 0$. Thus $S^\omega_n(\Phi) > 0$ implies $S^\omega_n(\Phi) \ge S_{nN}(\Phi)$. Since Φ_j is Diophantine for $1 \le j \le d$, there exists $c > 0$ such that $S_n(\Phi) > c^n$ for infinitely many $n \in \mathbb{N}$. By pigeonholing, there exists $0 \le l \le N-1$ such that $S_{nN+l}(\Phi) > c^{nN+l}$ for infinitely many $n \in \mathbb{N}$. Thus,

$$S_{nN}(\Phi) \ge S_{nN+l}(\Phi) > c^{nN+l} \ge (c^{2N})^n. \tag{8.10}$$

In what follows we let $\eta \in (0,1)$ and $n \in \mathbb{N}$ be with $\eta^{-1} \ll n$ such that (8.10) holds for n. Take M large enough so that $2r_{\max}^{MN} < c^{2N}$.

Let $\omega \in \Omega$. If $S^\omega_n(\Phi) = 0$, then $H(\nu^\omega_n, \mathcal{E}^\omega_{Mn}) = H(\beta^\omega, \mathcal{C}_{nN}) = 0$; If $S^\omega_n(\Phi) > 0$, then $S^\omega_n(\Phi) \ge S_{nN}(\Phi) > (c^{2N})^n$ by (8.10). From this, (4.2) and $2r_{\max}^{MN} < c^{2N}$, it follows that $H(\nu^\omega_n, \mathcal{E}^\omega_{Mn}) = H(\beta^\omega, \mathcal{C}_{nN})$. Hence,

$$H(\nu^\omega_n, \mathcal{E}^\omega_{Mn}) = H(\beta^\omega, \mathcal{C}_{nN}) \quad \text{for } \omega \in \Omega. \tag{8.11}$$

Let $\overline{\Omega}$ be the intersection of the $\overline{\Omega}$'s obtained from Lemma 6.1 with η, n in place of η, n, and Theorem 8.2 with η, n in place of ε, n. Then $\mathbf{P}(\overline{\Omega}) > 1 - O(\eta)$. For $\omega \in \overline{\Omega}$, we have

$$\begin{aligned}
N\kappa_{\mathcal{A}} &> \frac{1}{n} H(\mu^\omega, \mathcal{E}^\omega_n) - \eta && \text{(by Lemma 6.1)}\\
&> \frac{1}{n} H(\nu^\omega_n, \mathcal{E}^\omega_n) - O(\eta) && \text{(by Lemma 8.3)}\\
&> \frac{1}{n} H(\nu^\omega_n, \mathcal{E}^\omega_{Mn}) - O(\eta) && \text{(by Theorem 8.2)}\\
&= \frac{1}{n} H(\beta^\omega, \mathcal{C}_{nN}) - O(\eta). && \text{(by (8.11))}
\end{aligned}$$

Note that $\mathbf{P}(\overline{\Omega}) > 1 - O(\eta)$ and $H(\beta^\omega, \mathcal{C}_{nN})/(nN) \le H(p)$. From above, taking integral for ω in $\overline{\Omega}$ with respect to $\mathbf{P}$ gives

$$\begin{aligned}
\kappa_{\mathcal{A}} &\ge \int_{\overline{\Omega}} \frac{1}{nN} H(\beta^\omega, \mathcal{C}_{nN})\, \mathrm{d}\mathbf{P}(\omega) - O(\eta)\\
&\ge \int_{\Omega} \frac{1}{nN} H(\beta^\omega, \mathcal{C}_{nN})\, \mathrm{d}\mathbf{P}(\omega) - O(\eta)
\end{aligned}$$

$$= \frac{1}{nN} H\left(\beta, \mathcal{C}_{nN} \mid \widehat{\mathcal{A}}\right) - O(\eta) \qquad \text{(by (3.14))}$$
$$\geq h_{RW}(\Phi, \mathcal{A}) - O(\eta)\,. \qquad \text{(by (1.18))}$$

Letting $\eta \to 0$ shows that $\kappa_{\mathcal{A}} \geq h_{RW}(\Phi, \mathcal{A})$. Then by (6.1) and Lemma 8.5,

$$\dim \mathcal{A} \geq d - 1 + \frac{h_{RW}(\Phi, \mathcal{A}) - \sum_{j=1}^{d-1} \chi_j}{\chi_d} \geq \min \{d, f_\Phi(h_{RW}(\Phi, \mathcal{A}))\}\,.$$

This finishes the proof of the final case, and so Theorem 8.1. □

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