DIMENSION OF SELF-CONFORMAL MEASURES ASSOCIATED TO AN EXPONENTIALLY SEPARATED HOLOMORPHIC IFS

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Abstract. Let Φ be a holomorphic IFS on a bounded domain in $\mathbb{C}$. Suppose that the following conditions hold: (1) the maps in Φ do not have a common fixed point; (2) there does not exist a regular real-analytic curve which is invariant under all of the maps in Φ; (3) Φ is not holomorphically conjugate to a homothetic IFS; (4) Φ is exponentially separated. Under these assumptions, we show that the dimensions of the self-conformal measures associated to Φ, as well as the Hausdorff dimension of the associated self-conformal set, attain their natural upper bounds. The proof combines recently developed methods from the dimension theory of stationary fractal measures with complex-analytic arguments.

1. Introduction

1.1. **Background.** Let Ω be a bounded domain in $\mathbb{R}^d$. A function $f : \Omega \to \mathbb{R}^d$ is called conformal if it is smooth and, for each $x \in \Omega$, there exist $U_x \in \mathrm{O}(d)$ and $r_x > 0$ such that $f'(x) = r_x U_x$, where $\mathrm{O}(d)$ denotes the orthogonal group of $\mathbb{R}^d$. For $d = 1$, this is equivalent to f being smooth with nonvanishing derivative. For $d = 2$, upon identifying $\mathbb{R}^2$ with $\mathbb{C}$, it is equivalent to f being holomorphic or antiholomorphic with nonvanishing derivative. For $d \geq 3$, Liouville's theorem implies that f is conformal if and only if it is the restriction of a Möbius transformation.

Let Λ be a finite index set, and for each $i \in \Lambda$ let $\varphi_i : \overline{\Omega} \to \Omega$ be given. Assume that each φ_i extends to a conformal injection on a domain containing $\overline{\Omega}$ and that $0 < \|\varphi_i'(x)\|_{\mathrm{op}} < 1$ for all $i \in \Lambda$ and $x \in \overline{\Omega}$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. Setting $\Phi := \{\varphi_i\}_{i\in\Lambda}$, the collection Φ is called a conformal iterated function system (IFS) on Ω.

For a word $u = i_1...i_n \in \Lambda^n$, set $\varphi_u := \varphi_{i_1} \circ ... \circ \varphi_{i_n}$. For a sufficiently large $n \geq 1$, every map φ_u with $u \in \Lambda^n$ is a strict contraction of $\overline{\Omega}$. Hence, by [14], there exists a unique nonempty compact set $K_\Phi \subset \Omega$ satisfying $K_\Phi = \cup_{i\in\Lambda}\varphi_i(K_\Phi)$. This set is called the self-conformal set, or the attractor, associated with Φ.

Certain natural measures are also associated with Φ. Let $p = (p_i)_{i\in\Lambda}$ be a positive probability vector. By [14], there exists a unique Borel probability measure μ on Ω satisfying $\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\mu$, where $\varphi_i\mu$ denotes the pushforward of μ under

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φ_i. The measure μ, which is supported on K_Φ, is called the self-conformal measure corresponding to Φ and p.

By the work of Feng and Hu [8], the measure μ is exact dimensional. That is, there exists a number $\dim \mu$, called the dimension of μ, such that

$$\lim_{\delta \downarrow 0} \frac{\log \mu\left(B(x,\delta)\right)}{\log \delta} = \dim \mu \text{ for } \mu\text{-a.e. } x \in \Omega.$$

Here, $B(x,\delta)$ denotes the closed ball in $\mathbb{R}^d$ with centre x and radius δ. Computing the dimension of self-conformal measures is a natural and important problem in fractal geometry.

Write $H(p)$ for the entropy of p and $\chi := \chi(\Phi, p)$ for the Lyapunov exponent associated to Φ and p. That is,

$$H(p) := -\sum_{i\in\Lambda} p_i \log p_i \text{ and } \chi := -\sum_{i\in\Lambda} p_i \int \log \|\varphi_i'(x)\|_{\mathrm{op}}\, d\mu(x),$$

where we always use 2 as the base of the logarithm. It is easy to show that $H(p)/\chi$ is always an upper bound for $\dim \mu$. Moreover, in the absence of certain obvious obstructions, one generally expects that

$$\dim \mu = \min\left\{d, H(p)/\chi\right\}, \tag{1.1}$$

although this is often difficult to verify.

The IFS Φ is said to satisfy the strong separation condition (SSC) if the sets $\{\varphi_i(K_\Phi)\}_{i\in\Lambda}$ are disjoint, in which case it is easy to establish (1.1). Additionally, in various settings, this equality can be shown to hold almost surely under a natural randomization of the parameters (see, e.g., [26, Section 7] and [4, Chapter 14]). It is therefore desirable to find explicit conditions ensuring (1.1) that are much milder than the restrictive SSC.

The IFS Φ is called self-similar if each φ_i is a contracting similarity. That is, for each $i \in \Lambda$, there exist $0 < r_i < 1$, $U_i \in \mathrm{O}(d)$, and $a_i \in \mathbb{R}^d$ such that $\varphi_i(x) = r_i U_i x + a_i$ for all $x \in \Omega$. In recent years, major progress has been made on the above problem in the self-similar case, initiated by Hochman's seminal work [9] on the real line. Let $\|\cdot\|_\Omega$ denote the supremum norm on Ω. That is, $\|f\|_\Omega = \sup_{x\in\Omega} |f(x)|$ for a bounded function $f : \Omega \to \mathbb{R}^d$.

Definition 1.1. We say that Φ is exponentially separated if there exists $c > 0$ such that for infinitely many $n \in \mathbb{Z}_{>0}$,

$$\|\varphi_{u_1} - \varphi_{u_2}\|_\Omega \geq c^n \text{ for all distinct } u_1, u_2 \in \Lambda^n.$$

The exponential separation assumption is significantly weaker than both the SSC and the well-known open set condition, which we do not state here.

The main result of [9] states that (1.1) holds whenever $d = 1$ and Φ is self-similar and exponentially separated. Later, in [11], Hochman extended this result to higher dimensions. Kittle and Kogler [15] subsequently extended Hochman's higher-dimensional result to the contracting-on-average setting. In the uniformly contracting setting considered here, it follows directly from [11, Theorem 1.5] that (1.1) holds whenever the following conditions are all satisfied:

(A1) Φ is self-similar;

(A2) there is no proper affine subspace of $\mathbb{R}^d$ that is invariant under all the maps in Φ;

(A3) there is no nontrivial linear subspace of $\mathbb{R}^d$ that is invariant under all the linear parts of the maps in Φ;

(A4) Φ is exponentially separated.

It is a challenging and important open problem to relax these assumptions as much as possible.

Note that conditions (A2) and (A3) are necessary. Indeed, suppose that there exists a proper affine subspace V of $\mathbb{R}^d$ that is invariant under all the maps in Φ. Then $K_\Phi \subset V$ and hence $\mu(V) = 1$, which implies that $\dim \mu \leq \dim V$. Consequently, (1.1) fails whenever $H(p)/\chi > \dim V$. For the case $d = 1$, note that the condition (A2) is equivalent to requiring that the maps in Φ do not all share a common fixed point. This assumption is not stated explicitly in [9], since in the one-dimensional self-similar setting it follows from exponential separation.

Concerning the necessity of condition (A3), as demonstrated in [11, Example 1.2], one can construct exponentially separated self-similar measures μ for which condition (A2) holds but (1.1) fails. In these examples, μ is saturated, in a sense made precise in [11], along translates of a nontrivial linear subspace of $\mathbb{R}^d$ which is invariant under the linear parts of the maps in the IFS. Note that when $d = 2$ and the maps in Φ are orientation-preserving, the condition (A3) fails if and only if the maps in Φ are all homotheties. That is, for each $i \in \Lambda$, the map φ_i is of the form $\varphi_i(x) = r_i x + a_i$ for some $0 \neq r_i \in (-1, 1)$ and $a_i \in \mathbb{R}^d$.

Regarding the exponential separation assumption, note that it is necessary to assume some form of separation. Indeed, if Φ does not generate a free semigroup, in which case it is said to have exact overlaps, then (1.1) necessarily fails whenever $\dim \mu < d$. One of the most important open problems in fractal geometry, called the exact overlaps conjecture, asserts that in the self-similar setting on the real line, (1.1) can only fail in the presence of exact overlaps (see [10, 30] for further discussion). This conjecture has recently been verified in various situations (see [7, 19, 22, 29]). In higher dimensions, given the above result from [11], it is reasonable to expect that, when conditions (A1), (A2) and (A3) hold, (1.1) should hold whenever Φ generates a free semigroup.

It is easy to verify that Φ has exact overlaps if and only if there exist $n \geq 1$ and distinct $u_1, u_2 \in \Lambda^n$ such that $\varphi_{u_1} = \varphi_{u_2}$. Thus, exponential separation is a stronger quantitative version of the assumption that there are no exact overlaps.

Especially relevant to us is the extension of the above results beyond the self-similar setting. Hochman and Solomyak [13] proved that (1.1) holds for $d = 1$ when Φ is exponentially separated, its maps are restrictions of Möbius transformations, and the semigroup generated by Φ, viewed as a subsemigroup of $\mathrm{SL}(2, \mathbb{R})$, is strongly irreducible and proximal. Rapaport and Ren [21] obtained the analogous result for $d = 2$, with $\mathrm{SL}(2, \mathbb{C})$ in place of $\mathrm{SL}(2, \mathbb{R})$, under the additional requirement that there be no generalized circle invariant under all the maps in Φ[1].

In all the above results, the semigroup generated by Φ naturally embeds into a finite-dimensional Lie group, thereby avoiding significant difficulties that arise when this is not the case. In [20], Rapaport made substantial progress beyond this situation by proving that (1.1) holds for $d = 1$ when all the maps in Φ are real-analytic, have no common fixed point, and Φ is exponentially separated.

[1] The results of [13] and [21] are actually more general, as they do not require μ to be compactly supported.

It is natural to try to extend the preceding result to higher dimensions. As pointed out above, when $d \geq 3$, conformal maps are restrictions of Möbius transformations. Hence, in this case, self-conformal IFSs may be viewed naturally as finite subsets of the projective orthogonal group $\mathrm{PO}(d+1,1)$ (see, e.g., [25, Theorem 2.9]). Consequently, proving the desired equality for $d \geq 3$ should not require extending the arguments from [20] that were developed to handle the absence of an ambient finite-dimensional Lie group.

In the present paper, we extend the result of [20] to the case $d = 2$. Given a holomorphic IFS on a domain $\Omega \subset \mathbb{C}$, we establish (1.1) under mild assumptions that, apart from exponential separation, are necessary. This also extends the planar case of Hochman's self-similar result to the holomorphic setting.

1.2. **Setup and the main result.** Throughout the rest of the paper, let Ω be a bounded domain in $\mathbb{C}$, let Λ be a finite index set, and for each $i \in \Lambda$ let $\varphi_i : \overline{\Omega} \to \mathbb{C}$ be given. We shall always assume that

(i) there exists a domain $\Omega' \subset \mathbb{C}$ containing $\overline{\Omega}$ such that, for each $i \in \Lambda$, the map φ_i extends to a holomorphic injection on Ω';
(ii) for each $i \in \Lambda$, $\varphi_i\left(\overline{\Omega}\right) \subset \Omega$ and $0 < |\varphi_i'(z)| < 1$ for all $z \in \overline{\Omega}$.

Setting $\Phi := \{\varphi_i\}_{i \in \Lambda}$, the collection Φ is called a holomorphic IFS on Ω. Thus, viewing $\mathbb{C}$ as $\mathbb{R}^2$, every holomorphic IFS is in particular a conformal IFS.

As before, let K_Φ denote the self-conformal set associated with Φ. Fix a positive probability vector $p = (p_i)_{i \in \Lambda}$, and let μ denote the self-conformal measure associated with Φ and p. We continue to write $H(p)$ for the entropy of p and $\chi := \chi(\Phi, p)$ for the Lyapunov exponent associated with Φ and p.

To state our results, we need the following definitions.

Definition 1.2. Given an interval $J \subset \mathbb{R}$, a function $\gamma : J \to \mathbb{C}$ is said to be real-analytic if, for each $t_0 \in J$, there exist $\epsilon > 0$ and numbers $a_0, a_1, \ldots \in \mathbb{C}$ such that $\gamma(t) = \sum_{n \geq 0} a_n (t - t_0)^n$ for all $t \in J \cap (t_0 - \epsilon, t_0 + \epsilon)$.

Definition 1.3. A subset $\Gamma \subset \mathbb{C}$ is said to be a regular real-analytic curve if it is an embedded 1-dimensional real-analytic submanifold of $\mathbb{C}$. That is, for every $z_0 \in \Gamma$ there exist an open subset $U \subset \mathbb{C}$ containing z_0, an open interval $J \subset \mathbb{R}$, and a real-analytic map $\gamma : J \to \mathbb{C}$ such that $\gamma(J) = \Gamma \cap U$, γ is a homeomorphism onto $\Gamma \cap U$, and $\gamma'(t) \neq 0$ for all $t \in J$. When $\Gamma \subset \Omega$, we say that Γ is Φ-invariant if $\varphi_i(\Gamma) \subset \Gamma$ for each $i \in \Lambda$.

Remark. Note that a regular real-analytic curve need not be connected.

Definition 1.4. Given an open set $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is said to be homothetic if there exist $0 \neq r \in \mathbb{R}$ and $a \in \mathbb{C}$ such that $f(z) = rz + a$ for all $z \in U$. We say that Φ is holomorphically conjugate to a homothetic IFS if there exists an injective holomorphic map[2] $\psi : \Omega \to \mathbb{C}$ such that $\psi \circ \varphi_i \circ \psi^{-1} : \psi(\Omega) \to \mathbb{C}$ is homothetic for each $i \in \Lambda$.

Throughout the paper, we use exponential separation in the sense of Definition 1.1, identifying $\mathbb{C}$ with $\mathbb{R}^2$. The following theorem is our main result.

Theorem 1.5. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $\Phi = \{\varphi_i\}_{i \in \Lambda}$ be a holomorphic IFS on Ω (so that conditions (i) and (ii) hold). Suppose further that*

[2]Recall that an injective holomorphic map $\psi : \Omega \to \mathbb{C}$ is biholomorphic onto its image. That is, $\psi(\Omega)$ is open and $\psi^{-1} : \psi(\Omega) \to \Omega$ is also holomorphic; see, e.g., [28].

- *(M1) the maps in Φ do not have a common fixed point;*
- *(M2) there does not exist a regular real-analytic curve $\Gamma \subset \Omega$ which is Φ-invariant and closed in Ω;*
- *(M3) Φ is not holomorphically conjugate to a homothetic IFS;*
- *(M4) Φ is exponentially separated.*

Then for every positive probability vector $p = (p_i)_{i\in\Lambda}$,

$$\dim \mu = \min \{2, H(p)/\chi(\Phi, p)\}, \tag{1.2}$$

where μ is the self-conformal measure associated to Φ and p.

Let us make a few remarks on the assumptions appearing in the theorem. Conditions (i) and (ii) are standard in the context of conformal IFSs. Although condition (i) requires the maps in Φ to be holomorphic, our arguments also apply, with only minor modifications, when some of them are anti-holomorphic. For simplicity of exposition, we restrict to the holomorphic case. The injectivity assumption in condition (i) is needed to obtain certain bi-Lipschitz estimates (see Section 2.3) and to carry out various compactness arguments (see Lemma 3.5). Condition (ii) ensures that sufficiently long compositions of maps in Φ are uniformly contracting and that the bounded distortion property holds (see Lemma 2.2).

Assumption (M1) is equivalent to K_Φ not being a singleton. This is clearly necessary for the validity of (1.2), unless Φ consists of a single map.

Assumption (M2) also cannot be omitted in general. Indeed, if there exists a regular real-analytic curve $\Gamma \subset \Omega$ that is Φ-invariant and closed in Ω, then $K_\Phi \subset \Gamma$, and hence $\dim \mu \le 1$. Consequently, (1.2) fails whenever $H(p)/\chi > 1$. In the self-similar case, it is not difficult to see that assumptions (M1) and (M2) together are equivalent to Hochman's affine irreducibility condition (A2). Notice that assumption (M2) excludes only Φ-invariant curves that are closed in Ω. This slight strengthening of the theorem will be useful in subsequent applications of our main result.

Assumption (M3) is again necessary in general. Indeed, the example in [11, Example 1.2] mentioned above shows that the homothetic case must be excluded. Since holomorphic conjugacy leaves both $\dim \mu$ and χ unchanged, one must also exclude IFSs that are holomorphically conjugate to a homothetic IFS. In the holomorphic self-similar case, it is not difficult to see that assumption (M3) is equivalent to Hochman's linear irreducibility condition (A3).

Although assumptions (M2) and (M3) are necessary, they may be difficult to verify in concrete situations. In Corollary 1.7 below, we replace these two assumptions with a single condition that implies both and is often easier to verify.

Finally, as in the self-similar case, although exponential separation is a rather mild condition, it is desirable to weaken it to the necessary assumption that there are no exact overlaps. Since the exact overlaps conjecture remains open in full generality even for self-similar IFSs on the real line, this is currently well beyond our reach.

1.3. **Dimension of self-conformal sets.** Using Theorem 1.5, we can compute the dimension of the self-conformal set K_Φ. In what follows, $\dim_H$ denotes Hausdorff dimension.

For each $t \geq 0$ set

$$P_\Phi(t) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{u\in\Lambda^n} \|\varphi_u'\|_\Omega^t \right),$$

where $\|\cdot\|_\Omega$ denotes the supremum norm on Ω and the limit exists by sub-additivity. It is easy to verify that $P_\Phi(0) = \log|\Lambda|$, $P_\Phi(t) \to -\infty$ as $t \to \infty$, and P_Φ is strictly decreasing and continuous. Thus, there exists a unique $s(\Phi) \geq 0$ such that $P_\Phi(s(\Phi)) = 0$. Following [4], we call the number $s(\Phi)$ the conformal similarity dimension associated to Φ. It is well known and easy to show that $s(\Phi)$ is always an upper bound of $\dim_H K_\Phi$.

Corollary 1.6. *Let $\Omega \subset \mathbb{C}$ be a bounded domain, and let $\Phi = \{\varphi_i\}_{i\in\Lambda}$ be a holomorphic IFS on Ω satisfying the assumptions of Theorem 1.5. Then*

$$\dim_H K_\Phi = \min\{2, s(\Phi)\}, \tag{1.3}$$

where K_Φ denotes the self-conformal set associated to Φ.

The proof of Corollary 1.6 is given in Section 7.

1.4. **A verifiable sufficient criterion.** The following corollary provides a convenient sufficient criterion for applying Theorem 1.5 and Corollary 1.6. Let Λ^* denote the set of finite words over Λ.

Corollary 1.7. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and let $\Phi = \{\varphi_i\}_{i\in\Lambda}$ be a holomorphic IFS on Ω. Suppose that*

1. *the maps in Φ do not have a common fixed point;*
2. *there exists $u \in \Lambda^*$ such that $\varphi_u'(z_u) \notin \mathbb{R}$, where z_u denotes the unique fixed point of φ_u in Ω;*
3. *Φ is exponentially separated.*

Then $\dim_H K_\Phi = \min\{2, s(\Phi)\}$, where K_Φ is the self-conformal set associated to Φ. Moreover, for any positive probability vector $p = (p_i)_{i\in\Lambda}$, we have $\dim\mu = \min\left\{2, \frac{H(p)}{\chi(\Phi,p)}\right\}$, where μ is the self-conformal measure associated to Φ and p.

Proof. By Theorem 1.5 and Corollary 1.6, it suffices to verify assumptions (M2) and (M3) from Theorem 1.5.

Assume by contradiction that there exists a regular real-analytic curve $\Gamma \subset \Omega$ which is Φ-invariant and closed in Ω. Being Φ-invariant and closed in Ω, it is clear that Γ contains K_Φ. Moreover, it is clear that $z_u \in K_\Phi$, and so $z_u \in \Gamma$. Write V for the tangent space of Γ at z_u, so that V is a 1-dimensional real subspace of $\mathbb{C}$. Since φ_u is smooth on Ω and Γ is Φ-invariant and embedded in Ω, the restriction of φ_u to Γ is smooth as a map from the manifold Γ into itself. Hence, from $\varphi_u(z_u) = z_u$, we obtain that $\varphi_u'(z_u)V = V$. But this contradicts $\varphi_u'(z_u) \notin \mathbb{R}$, and so there cannot exist a regular real-analytic curve $\Gamma \subset \Omega$ which is Φ-invariant and closed in Ω.

Next, assume by contradiction that there exists a holomorphic and injective $\psi : \Omega \to \mathbb{C}$ such that $\psi \circ \varphi_i \circ \psi^{-1} : \psi(\Omega) \to \mathbb{C}$ is homothetic for each $i \in \Lambda$. It is clear that $h := \psi \circ \varphi_u \circ \psi^{-1}$ is homothetic, and so $h'(\psi(z_u)) \in \mathbb{R}$. On the other hand, by the chain rule and since $\varphi_u(z_u) = z_u$, we have $h'(\psi(z_u)) = \varphi_u'(z_u) \notin \mathbb{R}$. This contradiction shows that Φ is not holomorphically conjugate to a homothetic IFS, which completes the proof of the corollary. ∎

Remark. In the proof above, we showed that condition (2) appearing in Corollary 1.7 implies assumptions (M2) and (M3). It is unclear to us whether the converse holds, which seems to be an interesting question.

In the next subsection, we use Corollary 1.7 to show that the expected dimension formulas (1.2) and (1.3) hold for all parameters outside a set of Hausdorff dimension zero in the context of one-parameter families of holomorphic IFSs. By contrast, we do not pursue here the interesting problem of applying our results to obtain explicit non-self-similar examples for which these formulas hold. Nevertheless, we briefly comment on this direction.

In [3], Bárány, Kolossváry and Troscheit established a sufficient condition for an analytic IFS on $\mathbb{R}$ to be exponentially separated. Combining their result with the result of Rapaport [20] mentioned above yields explicit examples of overlapping, non-self-similar analytic IFSs on $\mathbb{R}$ whose associated self-conformal sets and measures satisfy the expected dimension formulas. It seems likely that the results of [3] could be extended to the holomorphic setting in a form that, when combined with Corollary 1.7, would yield explicit examples in our setting. We leave this as a problem for further research.

1.5. **One-parameter families of holomorphic IFSs.** In this subsection, we combine Corollary 1.7 with a result of Solomyak and Takahashi [27] to deduce a statement about one-parameter families of holomorphic IFSs.

Let $\Omega' \subset \mathbb{C}$ be a domain containing $\overline{\Omega}$, let J be a compact interval in $\mathbb{R}$, and suppose that the index set Λ has cardinality at least 2. For each $i \in \Lambda$, let $\{\varphi_{i,t}\}_{t\in J}$ be a one-parameter family of maps from Ω' into $\mathbb{C}$. Suppose that

(1) $\varphi_{i,t} : \Omega' \to \mathbb{C}$ is holomorphic and injective for all $i \in \Lambda$ and $t \in J$;
(2) $\varphi_{i,t}\left(\overline{\Omega}\right) \subset \Omega$ and $0 < \left|\varphi'_{i,t}(z)\right| < 1$ for all $i \in \Lambda$, $t \in J$ and $z \in \overline{\Omega}$;
(3) for each $i \in \Lambda$, the map $(t, z) \mapsto \varphi_{i,t}(z)$ extends to a real-analytic map on an open neighborhood of $J \times \Omega'$.

For $t \in J$, set $\Phi_t := \{\varphi_{i,t}\}_{i\in\Lambda}$ and let K_{Φ_t} be the self-conformal set corresponding to Φ_t. Given a probability vector $p = (p_i)_{i\in\Lambda}$, write $\mu_{t,p}$ for the self-conformal measure associated to Φ_t and p.

Fix some $z_0 \in \Omega$. Given $\omega, \eta \in \Lambda^{\mathbb{N}}$, let $F_{\omega,\eta} : J \to \mathbb{C}$ be defined by

$$F_{\omega,\eta}(t) := \lim_{n\to\infty} \left(\varphi_{\omega_0,t} \circ ... \circ \varphi_{\omega_n,t}(z_0) - \varphi_{\eta_0,t} \circ ... \circ \varphi_{\eta_n,t}(z_0)\right) \text{ for } t \in J,$$

where the existence of the limit follows easily from assumption (2). We say that the family of IFSs $\{\Phi_t\}_{t\in J}$ is real (resp. imaginary) non-degenerate if, for all distinct $\omega, \eta \in \Lambda^{\mathbb{N}}$, the function $\mathrm{Re} \circ F_{\omega,\eta}$ (resp. $\mathrm{Im} \circ F_{\omega,\eta}$) is not identically zero.

Given $i_1...i_n = u \in \Lambda^*$ and $t \in J$, set $\varphi_{u,t} := \varphi_{i_1,t} \circ ... \circ \varphi_{i_n,t}$ and write $z_{u,t}$ for the unique fixed point of $\varphi_{u,t}$.

Corollary 1.8. *Suppose that $\{\Phi_t\}_{t\in J}$ is either real non-degenerate or imaginary non-degenerate. Assume moreover that there exist $u \in \Lambda^*$ and $t_0 \in J$ such that $\varphi'_{u,t_0}(z_{u,t_0}) \notin \mathbb{R}$. Then there exists $E \subset J$ with $\dim_H E = 0$ such that, for all $t \in J \setminus E$, we have $\dim_H K_{\Phi_t} = \min\{2, s(\Phi_t)\}$ and $\dim \mu_{t,p} = \min\{2, H(p)/\chi(\Phi_t, p)\}$ for every positive probability vector $p = (p_i)_{i\in\Lambda}$.*

Proof. Let E_1 be the set of $t \in J$ for which the maps in Φ_t have a common fixed point. Let $u \in \Lambda^*$ and $t_0 \in J$ be as in the statement of the corollary, and denote by E_2 the set of $t \in J$ for which $\varphi'_{u,t}(z_{u,t}) \in \mathbb{R}$. Additionally, write E_3 for the set of $t \in J$ for which Φ_t is not exponentially separated. By Corollary 1.7, in order to prove the corollary, it suffices to show that the sets E_1, E_2 and E_3 are all of Hausdorff dimension zero.

Since $\{\Phi_t\}_{t\in J}$ is real or imaginary non-degenerate, it follows from [27, Theorem 2.10] that $\dim_H E_3 = 0$. We next complete the proof by showing that E_1 and E_2 are finite.

Let $i_1, i_2 \in \Lambda$ be distinct. For $j = 1, 2$, let i_j^∞ denote the infinite word in $\Lambda^{\mathbb{N}}$ consisting only of the letter i_j, and set $F := F_{i_1^\infty, i_2^\infty}$. Given $t \in J$, note that $\varphi_{i_1,t}$ and $\varphi_{i_2,t}$ share a common fixed point if and only if $F(t) = 0$. Consequently, we have $E_1 \subset F^{-1}\{0\}$. Since $\{\Phi_t\}_{t\in J}$ is real or imaginary non-degenerate, F is not identically zero. Moreover, by [27, Lemma 2.6], F is real analytic on J. Together, these two facts imply that $F^{-1}\{0\}$ is finite, which shows that E_1 is finite.

Let $f : J \to \mathbb{R}$ be defined by $f(t) = \operatorname{Im}\left(\varphi'_{u,t}(z_{u,t})\right)$ for $t \in J$. Since $\varphi'_{u,t_0}(z_{u,t_0}) \notin \mathbb{R}$, the function f is not identically zero. Note that for each $t \in J$, we have $z_{u,t} = \lim_{n\to\infty} \varphi_{u,t}^n(z_0)$, where $\varphi_{u,t}^n$ denotes the composition of $\varphi_{u,t}$ with itself n times. Thus, by [27, Lemma 2.6], the function $t \mapsto z_{u,t}$ is real-analytic on J. Additionally, by our assumptions on $\{\Phi_t\}_{t\in J}$, the function $(t,z) \mapsto \varphi'_{u,t}(z)$ is real analytic on $J \times \Omega$. Combining these facts, we obtain that f is real analytic on J. Since f is not identically zero and $E_2 = f^{-1}\{0\}$, it follows that E_2 is finite, which completes the proof. □

1.6. **An overview of the proof.** In this subsection, we provide a brief overview of the proof of Theorem 1.5. For the sake of presentation, we allow ourselves to be rather imprecise and informal. All the arguments will be carried out rigorously and in detail in later sections of the paper.

We continue to use the notation introduced in Section 1.2, and suppose that Φ satisfies the assumptions of Theorem 1.5. Denote by $\mathcal{O}(\Omega)$ the vector space of holomorphic functions from Ω to $\mathbb{C}$. For $k \geq 1$, denote by $\mathcal{P}_k$ the vector space of polynomials $q \in \mathbb{C}[X]$ with $\deg q \leq k$. Given a set Y, write $\mathcal{M}_{\mathrm{fin}}(Y)$ for the collection of finitely supported probability measures on Y.

As in many other recent developments in the dimension theory of stationary fractal measures, the basic strategy of our proof is to extend Hochman's argument from [9], developed there in the self-similar setting on $\mathbb{R}$, to the setting studied here. In carrying out this strategy, we assume for contradiction that

$$\Delta := \min\{2, H(p)/\chi\} - \dim \mu > 0$$

and encounter convolutions of the form $\nu.\mu$, where $\nu \in \mathcal{M}_{\mathrm{fin}}(\mathcal{O}(\Omega))$ has non-negligible Shannon entropy in a way that depends on Δ[3]. Here, $\nu.\mu$ denotes the pushforward of $\nu \times \mu$ via the map $(f,z) \mapsto f(z)$. In order to successfully carry out the aforementioned extension of the argument from [9], we need to show that, in a certain sense to be made precise, the entropy of $\nu.\mu$ is significantly larger than that of μ alone.

At this point, we encounter major difficulties caused by the infinite-dimensionality of $\mathcal{O}(\Omega)$. In order to overcome these difficulties, we extend to our setting a method developed in [20] in the context of real-analytic systems on $\mathbb{R}$. This method enables us to reduce the above problem to a finite-dimensional situation. More precisely, it enables us to approximate the entropy of convolutions of the

[3]During the proof, we actually need to consider more general convolutions of the form $\nu.(\psi\mu)$, where $\psi : \Omega \to \mathbb{C}$ is a suitably controlled biholomorphism onto its image.

form $\nu.\mu$, with $\nu \in \mathcal{M}_{\text{fin}}(\mathcal{O}(\Omega))$, by the entropy of convolutions of the form $\nu'.\mu$, with $\nu' \in \mathcal{M}_{\text{fin}}(\mathcal{P}_k)$. Here, $k \geq 1$ is an integer which is assumed to be large with respect to Δ and the exponential separation constant c appearing in Definition 1.1.

In order to proceed with the overview of the proof, we introduce some additional notation and terminology. For $n \geq 0$, denote by $\mathcal{D}_n$ the dyadic partition of $\mathbb{C}$. Given $D \in \mathcal{D}_n$ with $\mu(D) > 0$, we call a measure of the form μ_D a component of μ. Very roughly speaking, given a line $\ell \in \mathbb{RP}^1$, the component μ_D is said to be saturated in the direction $\ell^\perp$ if the entropy of $\pi_\ell \mu_D$, where π_ℓ denotes the orthogonal projection onto ℓ, is close to its trivial lower bound, namely, the entropy of μ_D minus 1.

Given the above reduction, we shall need to establish an entropy increase result for convolutions $\nu'.\mu$, where $\nu' \in \mathcal{M}_{\text{fin}}(\mathcal{P}_k)$ has non-negligible entropy[4]. The proof of this result is based on Hochman's inverse theorem [11] for the entropy growth of convolutions of measures on $\mathbb{R}^d$. In order to apply this theorem, we shall need to show that most components of μ are not saturated in any direction, which is the main challenge in our proof of the entropy increase result.

To address this challenge, we shall extend to our setting a method from the work of Rapaport and Ren [21], which was used there to establish non-saturation of components in the context of Furstenberg measures on $\mathbb{CP}^1$. This method relies on an ergodic-theoretic argument involving a certain cocycle, which, following [21], we call the direction cocycle. A key step is to show that the direction cocycle is not a coboundary. The proof of this fact relies on extending ideas from [21] and combining them with complex-analytic techniques.

Structure of the paper. In Section 2, we introduce the necessary notation and establish some basic facts that will be used throughout the paper. In Section 3, we show that μ vanishes on real-analytic curves, which is used in several places in our arguments. In Section 4, we establish the aforementioned non-saturation property. In Section 5, we prove the aforementioned entropy increase result. In Section 6, we prove our main result, Theorem 1.5. Finally, in Section 7, we prove Corollary 1.6, concerning the dimension of self-conformal sets.

2. Preliminaries

2.1. **Basic notation.** Throughout the paper, the base of the logarithm is 2.

For a metric space X, denote by $\mathcal{M}(X)$ the collection of all compactly supported Borel probability measures on X. Given another metric space Y, a Borel map $f : X \to Y$, and a measure $\nu \in \mathcal{M}(X)$, we write $f\nu := \nu \circ f^{-1}$ for the pushforward of ν via f. For a Borel set $E \subset X$ with $\nu(E) > 0$, we denote by ν_E the conditioning of ν on E. That is, $\nu_E := \frac{1}{\nu(E)}\nu|_E$, where $\nu|_E$ is the restriction of ν to E.

Given a partition $\mathcal{D}$ of a set X, for $x \in X$ we denote by $\mathcal{D}(x)$ the unique $D \in \mathcal{D}$ containing x.

Given an integer $n \geq 1$, let $\mathcal{N}_n := \{1, ..., n\}$, and denote the normalized counting measure on $\mathcal{N}_n$ by λ_n; that is, $\lambda_n\{i\} = 1/n$ for each $1 \leq i \leq n$.

For a complex vector space V, a scalar $c \in \mathbb{C}$, and a vector $v \in V$, define $S_c : V \to V$ and $T_v : V \to V$ by $S_c w = cw$ and $T_v w = v + w$ for $w \in V$. We will often use this notation with $V = \mathbb{C}$.

[4]We shall actually prove such a statement for compactly supported probability measures ν' on $\mathcal{P}_k$, rather than just for finitely supported probability measures.

Given $z \in \mathbb{C}$ and $r > 0$, denote by $D_r(z)$, $\overline{D}_r(z)$ and $C_r(z)$ the open disc, closed disc and circle in $\mathbb{C}$ with centre z and radius r. Write $\mathbb{D} := D_1(0)$ for the open unit disc. For a nonempty subset $E \subset \mathbb{C}$, we denote by $E^{(r)}$ the r-thickening of E; that is,

$$E^{(r)} := \{z \in \mathbb{C} \,:\, |z - w| \le r \text{ for some } w \in E\}.$$

We denote by $\mathbb{RP}^1$ the set of real lines through the origin in $\mathbb{C}$; that is, $\mathbb{RP}^1 := \{z\mathbb{R} \,:\, 0 \ne z \in \mathbb{C}\}$. For $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$, we set $z\mathbb{R}w\mathbb{R} := zw\mathbb{R}$, which makes $\mathbb{RP}^1$ into a multiplicative group whose identity element is $\mathbb{R}$. We denote by $(z\mathbb{R})^\perp \in \mathbb{RP}^1$ the line perpendicular to $z\mathbb{R}$.

For $z, w \in \mathbb{C}$ with $|z| = |w| = 1$, write

$$d\,(z\mathbb{R}, w\mathbb{R}) := \left(1 - (\mathrm{Re}\,(z\overline{w}))^2\right)^{1/2},$$

which defines a metric on $\mathbb{RP}^1$. In fact, $d\,(z\mathbb{R}, w\mathbb{R})$ equals the modulus of the sine of the angle between $z\mathbb{R}$ and $w\mathbb{R}$. Given $r > 0$, we write $B\,(z\mathbb{R}, r)$ for the closed ball in $\left(\mathbb{RP}^1, d\right)$ with center $z\mathbb{R}$ and radius r.

For $z\mathbb{R} \in \mathbb{RP}^1$, we denote by $\pi_{z\mathbb{R}} : \mathbb{C} \to \mathbb{C}$ the orthogonal projection onto $z\mathbb{R}$, where $\mathbb{C}$ is identified with $\mathbb{R}^2$; that is,

$$\pi_{z\mathbb{R}}(w) = |z|^{-2}\mathrm{Re}\,(w\overline{z})\,z \quad \text{for } w \in \mathbb{C}.$$

Relations between parameters. Given $R_1, R_2 \in \mathbb{R}$ with $R_1, R_2 \ge 1$, we write $R_1 \ll R_2$ in order to indicate that R_2 is large with respect to R_1. Formally, this means that $R_2 \ge f(R_1)$, where f is an unspecified function from $[1, \infty)$ into itself. The values attained by f are assumed to be sufficiently large in a manner depending on the specific context.

Similarly, given $0 < \epsilon_1, \epsilon_2 < 1$ we write $R_1 \ll \epsilon_1^{-1}$, $\epsilon_2^{-1} \ll R_2$ and $\epsilon_1^{-1} \ll \epsilon_2^{-1}$ in order to respectively indicate that ϵ_1 is small with respect to R_1, R_2 is large with respect to ϵ_2, and ϵ_2 is small with respect to ϵ_1.

The relation $\ll$ is clearly transitive. That is, if $R_1 \ll R_2$ and for $R_3 \ge 1$ we have $R_2 \ll R_3$, then also $R_1 \ll R_3$. For instance, the sentence 'Let $m \ge 1$, $k \ge K(m) \ge 1$ and $n \ge N(m, k) \ge 1$ be given' is equivalent to 'Let $m, k, n \ge 1$ be with $m \ll k \ll n$'.

2.2. **The setup.** As in Section 1.2, let $\Omega \subset \mathbb{C}$ be a bounded domain, let Λ be a finite index set, and let $\Phi = \{\varphi_i\}_{i \in \Lambda}$ be a holomorphic IFS on Ω. In what follows, we always assume that Φ satisfies all the assumptions in Theorem 1.5, except the exponential separation condition, which is assumed only in Sections 6 and 7, where we prove our main results. Without loss of generality, suppose that $0 \in \Omega$.

Throughout the paper, we fix a bounded domain $\Omega_0 \subset \mathbb{C}$, containing $\overline{\Omega}$, such that the maps φ_i are defined on $\overline{\Omega_0}$ and conditions (i) and (ii) hold with Ω_0 in place of Ω.

Let $K_\Phi \subset \Omega$ denote the self-conformal set associated to Φ. That is, K_Φ is the unique nonempty compact subset of Ω such that $K_\Phi = \cup_{i \in \Lambda} \varphi_i(K_\Phi)$.

Fix a positive probability vector $p = (p_i)_{i \in \Lambda}$, and let μ denote the self-conformal measure associated to Φ and p. That is, μ is the unique element in $\mathcal{M}(\Omega)$ satisfying the relation

$$\mu = \sum_{i \in \Lambda} p_i \cdot \varphi_i \mu. \tag{2.1}$$

Note that the support of μ is K_Φ.

As above, let $\chi := \chi(\Phi, p)$ denote the Lyapunov exponent associated to Φ and p. That is,

$$\chi := -\sum_{i\in\Lambda} p_i \int \log|\varphi_i'(z)| \, d\mu(z). \tag{2.2}$$

2.3. **A bi-Lipschitz estimate.** Given $0 < \epsilon \le 1$, let $\mathcal{F}_\epsilon$ denote the set of holomorphic injective maps $\psi : \Omega_0 \to \mathbb{C}$ satisfying $\epsilon \le |\psi'(z)| \le \epsilon^{-1}$ for all $z \in \Omega_0$.

Lemma 2.1. *There exists a constant $C > 1$ such that for all $0 < \epsilon \le 1$ and $\psi \in \mathcal{F}_\epsilon$,*

$$C^{-1}\epsilon|z-w| \le |\psi(z) - \psi(w)| \le C\epsilon^{-1}|z-w| \text{ for all } z, w \in \Omega.$$

Proof. Let $C > 1$ be large with respect to Ω and Ω_0, and let $0 < \epsilon \le 1$ and $\psi \in \mathcal{F}_\epsilon$ be given.

We first prove the upper bound. For $z, w \in \Omega$, write $d(z,w)$ for the infimum of the lengths of smooth paths $\gamma : [0,1] \to \Omega_0$ with $\gamma(0) = z$ and $\gamma(1) = w$. As shown in the proof of [18, Lemma 2.1], assuming C is sufficiently large we have

$$d(z,w) \le C|z-w| \text{ for all } z, w \in \Omega.$$

Given $z, w \in \Omega$ and $\delta > 0$, there exists a smooth path $\gamma : [0,1] \to \Omega_0$ such that $\gamma(0) = z$, $\gamma(1) = w$, and $\text{length}(\gamma) \le d(z,w) + \delta$. Note that $\psi \circ \gamma$ is a smooth path from $\psi(z)$ to $\psi(w)$. Hence,

$$\begin{aligned}|\psi(z) - \psi(w)| \le \text{length}(\psi\circ\gamma) &= \int_0^1 |\psi'(\gamma(t))| \cdot |\gamma'(t)| \, dt \le \epsilon^{-1} \int_0^1 |\gamma'(t)| \, dt \\ &= \epsilon^{-1}\text{length}(\gamma) \le \epsilon^{-1}(d(z,w) + \delta) \le \epsilon^{-1}(C|z-w| + \delta).\end{aligned}$$

Since this holds for all $\delta > 0$, the upper bound follows.

We turn to the proof of the lower bound. We may assume that $D_{1/C}(z) \subset \Omega_0$ for all $z \in \Omega$. Let $z, w \in \Omega$, and set $r := \min\{1/C, |z-w|\}$. By the Koebe quarter theorem (see, e.g., [23, Theorem 14.14]),

$$D\left(\psi(z), \frac{1}{4}|\psi'(z)|\, r\right) \subset \psi(D(z,r)).$$

Thus, since $w \notin D(z,r)$ and ψ is injective on Ω_0,

$$|\psi(z) - \psi(w)| \ge \frac{1}{4}|\psi'(z)|\, r \ge \frac{\epsilon r}{4}.$$

Since Ω is bounded, we may assume that $r \ge |z-w|/C^2$. Hence,

$$|\psi(z) - \psi(w)| \ge \frac{\epsilon}{4C^2}|z-w|,$$

which completes the proof of the lemma. ∎

2.4. **Bounded distortion and related auxiliary results.** Recall that Ω_0 is a bounded domain containing $\overline{\Omega}$, the maps φ_i are defined on $\overline{\Omega_0}$, and conditions (i) and (ii) hold with Ω_0 in place of Ω. This implies that Φ satisfies the so-called bounded distortion property on Ω_0, as stated in the following lemma. Its proof can be found, for example, in [18, Lemma 2.1]. Recall that Λ^* denotes the set of finite words over Λ.

Lemma 2.2 (Bounded distortion property)**.** *There exists a constant $C \ge 1$ such that $|\varphi_u'(z)| \le C\,|\varphi_u'(w)|$ for all $u \in \Lambda^*$ and $z, w \in \Omega_0$.*

Combining the bounded distortion property with Lemma 2.1 we obtain the following.

Lemma 2.3. *There exists a constant $C > 1$ such that for all $u \in \Lambda^*$, $0 < \epsilon \le 1$, $\psi \in \mathcal{F}_\epsilon$, and $z, w \in \Omega$*

$$C^{-1}\epsilon\,|\varphi_u'(0)|\,|z-w| \le |\psi\circ\varphi_u(z) - \psi\circ\varphi_u(w)| \le C\epsilon^{-1}\,|\varphi_u'(0)|\,|z-w|.$$

Proof. Let $C > 1$ be a constant at least as large as the constants obtained in Lemmas 2.1 and 2.2. Let $u \in \Lambda^*$, $0 < \epsilon \le 1$ and $\psi \in \mathcal{F}_\epsilon$ be given. By the chain rule and Lemma 2.2, $S^{-1}_{\varphi_u'(0)} \circ \psi \circ \varphi_u \in \mathcal{F}_{\epsilon/C}$. Hence, by Lemma 2.1, for each $z, w \in \Omega$

$$C^{-2}\epsilon|z-w| \le \left| S^{-1}_{\varphi_u'(0)} \circ \psi \circ \varphi_u(z) - S^{-1}_{\varphi_u'(0)} \circ \psi \circ \varphi_u(w) \right| \le C^2\epsilon^{-1}|z-w|,$$

which proves the lemma. □

Combining Cauchy's estimates with the above considerations, we can also obtain the following estimate for the higher derivatives.

Lemma 2.4. *There exists a constant $C > 1$ such that for all $u \in \Lambda^*$, $0 < \epsilon \le 1$, $\psi \in \mathcal{F}_\epsilon$, $k \in \mathbb{Z}_{>0}$, and $z \in \Omega$,*

$$\left| (\psi\circ\varphi_u)^{(k)}(z) \right| \le k!C^k\epsilon^{-1}\,|\varphi_u'(0)|\,,$$

where $(\psi\circ\varphi_u)^{(k)}$ denotes the k-th derivative of $\psi\circ\varphi_u$.

Proof. Let $0 < \delta < 1$ be sufficiently small so that the closure of $\Omega^{(\delta)}$ is contained in Ω_0. By the arguments given in the proofs of Lemmas 2.1 and 2.3, there exists $C = C(\delta) > 1$ such that for all $u \in \Lambda^*$, $0 < \epsilon \le 1$ and $\psi \in \mathcal{F}_\epsilon$,

$$|\psi\circ\varphi_u(z) - \psi\circ\varphi_u(w)| \le C\epsilon^{-1}\,|\varphi_u'(0)|\,|z-w| \text{ for all } z, w \in \Omega^{(\delta)}. \tag{2.3}$$

Let $u \in \Lambda^*$, $0 < \epsilon \le 1$ and $\psi \in \mathcal{F}_\epsilon$ be given, and set $f := \psi\circ\varphi_u - \psi\circ\varphi_u(0)$. From (2.3) and by assuming C is sufficiently large, for $w \in \Omega^{(\delta)}$

$$|f(w)| \le C\epsilon^{-1}\,|\varphi_u'(0)|\,|w| \le C^2\epsilon^{-1}\,|\varphi_u'(0)|\,.$$

Thus, by Cauchy's estimate, for $k \in \mathbb{Z}_{>0}$ and $z \in \Omega$

$$\left| (\psi\circ\varphi_u)^{(k)}(z) \right| = \left| f^{(k)}(z) \right| \le k!\delta^{-k}\cdot C^2\epsilon^{-1}\,|\varphi_u'(0)|\,,$$

which completes the proof of the lemma. □

2.5. **Function spaces and Taylor approximation.** Denote by $\mathcal{O}(\Omega)$ the vector space of holomorphic functions from Ω to $\mathbb{C}$.

Given $k \ge 1$, denote by $\mathcal{P}_k$ the vector space of polynomials $q \in \mathbb{C}[X]$ with $\deg q \le k$. Let $\|\cdot\|_2$ be the norm on $\mathcal{P}_k$ defined by

$$\|q\|_2^2 := \sum_{j=0}^k |a_j|^2 \text{ for } q(X) = \sum_{j=0}^k a_j X^j \in \mathcal{P}_k.$$

In what follows, all metric and topological concepts in $\mathcal{P}_k$ are considered with respect to $\|\cdot\|_2$.

Let ν be a finitely supported probability measure on $\mathcal{O}(\Omega)$ or an element of $\mathcal{M}(\mathcal{P}_k)$ for some $k \ge 1$. Given $\theta \in \mathcal{M}(\Omega)$, we write $\nu.\theta$ for the convolution of ν with θ. That is, $\nu.\theta$ denotes the pushforward of $\nu \times \theta$ via the map sending

$(f,z) \in \mathcal{O}(\Omega) \times \Omega$ to $f(z)$. For $z \in \Omega$ we write $\nu.z$ in place of $\nu.\delta_z$, where δ_z is the Dirac mass at z.

For $k \geq 1$ and $a \in \Omega$, let $P_{k,a} : \mathcal{O}(\Omega) \to \mathcal{P}_k$ be the linear map such that $P_{k,a}f$ equals the k-th order Taylor polynomial of $f \in \mathcal{O}(\Omega)$ at the point a. That is,

$$P_{k,a}f(X) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(X-a)^j. \tag{2.4}$$

The following Taylor estimate will be crucial for our arguments.

Lemma 2.5. *There exist $\delta > 0$ and $C > 1$ such that for all $u \in \Lambda^*$, $0 < \epsilon \leq 1$, $\psi \in \mathcal{F}_\epsilon$, $k \geq 1$, and $a, z \in \Omega$ with $|z-a| < \delta$,*

$$|\psi \circ \varphi_u(z) - P_{k,a}(\psi \circ \varphi_u)(z)| \leq C^{k+1}\epsilon^{-1}|\varphi_u'(0)|\,|z-a|^{k+1}.$$

Proof. Let $\delta > 0$ be sufficiently small so that $\Omega^{(\delta)} \subset \Omega_0$. Let $C > 1$ be the constant obtained in Lemma 2.4, and let $u \in \Lambda^*$, $0 < \epsilon \leq 1$, $\psi \in \mathcal{F}_\epsilon$, $k \geq 1$, and $a, z \in \Omega$ with $0 < |z-a| < \min\{\delta, 1/(2C)\}$. Since $\psi \circ \varphi_u$ is analytic on Ω_0 and $z \in D_\delta(a) \subset \Omega_0$,

$$\psi \circ \varphi_u(z) = \sum_{j=0}^{\infty} \frac{(\psi \circ \varphi_u)^{(j)}(a)}{j!}(z-a)^j.$$

Thus, by Lemma 2.4 and since $C|z-a| < 1/2$,

$$\begin{aligned}
|\psi \circ \varphi_u(z) - P_{k,a}(\psi \circ \varphi_u)(z)| &\leq \sum_{j=k+1}^{\infty} \frac{\left|(\psi \circ \varphi_u)^{(j)}(a)\right|}{j!}|z-a|^j \\
&\leq 2C^{k+1}\epsilon^{-1}|\varphi_u'(0)|\,|z-a|^{k+1},
\end{aligned}$$

which completes the proof of the lemma. $\square$

2.6. **Entropy.** Let $(X, \mathcal{B})$ be a measurable space. Given a probability measure θ on X and a countable partition $\mathcal{D} \subset \mathcal{B}$ of X, the entropy of θ with respect to $\mathcal{D}$ is defined by

$$H(\theta, \mathcal{D}) := -\sum_{D \in \mathcal{D}} \theta(D) \log \theta(D).$$

If $\mathcal{E} \subset \mathcal{B}$ is another countable partition of X, the conditional entropy given $\mathcal{E}$ is defined as follows

$$H(\theta, \mathcal{D} \mid \mathcal{E}) := \sum_{E \in \mathcal{E}} \theta(E) \cdot H(\theta_E, \mathcal{D}).$$

Throughout the paper, we repeatedly use basic properties of entropy and conditional entropy. We frequently do so without explicit reference. Readers are advised to consult [9, Section 3.1] for details.

2.7. **Dyadic partitions and component measures.** For $d \geq 1$ and $n \geq 0$, denote by $\mathcal{D}_n^{\mathbb{C}^d}$ the level-n dyadic partition of $\mathbb{C}^d$, where $\mathbb{C}^d$ is identified with $\mathbb{R}^{2d}$. For a real number $t \geq 0$, we write $\mathcal{D}_t^{\mathbb{C}^d}$ in place of $\mathcal{D}_{\lfloor t \rfloor}^{\mathbb{C}^d}$, where $\lfloor t \rfloor$ denotes the integer part of t. We omit the superscript $\mathbb{C}^d$ when it is clear from the context.

Dyadic entropy, i.e. entropy measured with respect to the partitions $\mathcal{D}_n^{\mathbb{C}}$, plays an important role in our arguments. Note that by [31], when $\theta \in \mathcal{M}(\mathbb{C})$ is exact dimensional,

$$\lim_{n \to \infty} \frac{1}{n} H(\theta, \mathcal{D}_n) = \dim \theta. \tag{2.5}$$

We shall also need the following two simple lemmas. They follow easily from basic properties of entropy, and their proofs are therefore omitted.

Lemma 2.6. *Let $d \geq 1$ and $\theta \in \mathcal{M}\left(\mathbb{C}^d\right)$ be given. Let $C > 1$, and let $f : \operatorname{supp}(\theta) \to \mathbb{C}^d$ be bi-Lipschitz with bi-Lipschitz constant C. That is, $C^{-1}|x-y| \leq |f(x)-f(y)| \leq C|x-y|$ for $x, y \in \operatorname{supp}(\theta)$. Then for each $n \geq 0$,*

$$H\left(f\theta, \mathcal{D}_n\right) = H\left(\theta, \mathcal{D}_n\right) + O_d\left(1 + \log C\right).$$

Lemma 2.7. *Let $(X, \mathcal{B}, \theta)$ be a probability space, and let $f, g : X \to \mathbb{C}$ be measurable. Let $n \geq 0$, and suppose that $|f(x) - g(x)| \leq 2^{-n}$ for all $x \in X$. Then*

$$H\left(f\theta, \mathcal{D}_n\right) = H\left(g\theta, \mathcal{D}_n\right) + O(1).$$

We also need to consider dyadic partitions of vector spaces of polynomials. For $k \geq 1$ and $n \geq 0$, we denote by $\mathcal{D}_n^{\mathcal{P}_k}$ the level-n dyadic partition of $\mathcal{P}_k$, which is given by

$$\mathcal{D}_n^{\mathcal{P}_k} := \left\{ \left\{ \sum_{j=0}^{k} a_j X^j \ : \ (a_0, ..., a_k) \in D \right\} \ : \ D \in \mathcal{D}_n^{\mathbb{C}^{k+1}} \right\}.$$

Thus, $\mathcal{D}_n^{\mathcal{P}_k}$ is defined by identifying $\mathcal{P}_k$ with $\mathbb{C}^{k+1}$ in a natural way. We omit the superscript $\mathcal{P}_k$ when it is clear from the context.

Given $\theta \in \mathcal{M}(\mathbb{C})$, $n \geq 0$, and $z \in \mathbb{C}$ with $\theta\left(\mathcal{D}_n(z)\right) > 0$, we write $\theta_{z,n}$ in place of the conditional measure $\theta_{\mathcal{D}_n(z)}$. Similarly, for $\nu \in \mathcal{M}\left(\mathcal{P}_k\right)$ and $q \in \mathcal{P}_k$ with $\nu\left(\mathcal{D}_n(q)\right) > 0$, we write $\nu_{q,n}$ in place of $\nu_{\mathcal{D}_n(q)}$. The measures $\theta_{z,n}$ and $\nu_{q,n}$ are said to be level-n components of θ and ν respectively.

Throughout the rest of the paper, we use the probabilistic notations introduced in [9, Section 2.2]; readers are encouraged to consult this reference for further details. In particular, we often consider $\theta_{z,n}$ and $\nu_{q,n}$ as random measures in a natural way.

2.8. **Symbolic preliminaries.** Let $\Lambda^{\mathbb{N}}$ denote the set of one-sided infinite words over Λ. We equip $\Lambda^{\mathbb{N}}$ with the product topology, where each copy of Λ is equipped with the discrete topology. Let $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ denote the left-shift map. That is, $\sigma(\omega) = (\omega_{n+1})_{n \geq 0}$ for $(\omega_n)_{n \geq 0} = \omega \in \Lambda^{\mathbb{N}}$.

Given $n \geq 1$ and $(\omega_k)_{k \geq 0} = \omega \in \Lambda^{\mathbb{N}}$, write $\omega|_n := \omega_0 ... \omega_{n-1} \in \Lambda^n$ for the nth prefix of ω. For a word $u \in \Lambda^n$, denote by $[u] \subset \Lambda^{\mathbb{N}}$ the cylinder set determined by u. That is, $[u] := \left\{\omega \in \Lambda^{\mathbb{N}} \ : \ \omega|_n = u\right\}$. Let $\mathcal{C}_n := \{[u]\}_{u \in \Lambda^n}$ denote the partition of $\Lambda^{\mathbb{N}}$ into level-n cylinder sets. Given a set of words $\mathcal{U} \subset \Lambda^*$, we often write $[\mathcal{U}]$ in place of $\cup_{u \in \mathcal{U}} [u]$.

Denote by $\beta := p^{\mathbb{N}}$ the Bernoulli measure on $\Lambda^{\mathbb{N}}$ associated to p. That is, β is the unique element in $\mathcal{M}\left(\Lambda^{\mathbb{N}}\right)$ such that $\beta\left([u]\right) = p_u$ for $u \in \Lambda^*$, where $p_u := p_{i_1} \cdot ... \cdot p_{i_n}$ for $u = i_1 ... i_n \in \Lambda^n$.

Let $\Pi : \Lambda^{\mathbb{N}} \to \Omega$ be the coding map associated to Φ. That is,

$$\Pi(\omega) := \lim_{n \to \infty} \varphi_{\omega|_n}(0) \text{ for } \omega \in \Lambda^{\mathbb{N}},$$

where recall that we assumed that $0 \in \Omega$. Note that $\mu = \Pi\beta$. For $n \geq 1$, let $\Pi_n : \Lambda^{\mathbb{N}} \to \mathcal{O}(\Omega)$ be defined by $\Pi_n(\omega) = \varphi_{\omega|_n}$ for $\omega \in \Lambda^{\mathbb{N}}$.

From the ergodicity of $\left(\Lambda^{\mathbb{N}}, \sigma, \beta\right)$, by bounded distortion, and since $\mu = \Pi\beta$, it follows easily that

$$\chi = -\lim_{n \to \infty} \frac{1}{n} \log \left|\varphi'_{\omega|_n}(0)\right| \text{ for } \beta\text{-a.e. } \omega. \tag{2.6}$$

Write $\{\beta_\omega\}_{\omega\in\Lambda^{\mathbb{N}}} \subset \mathcal{M}\left(\Lambda^{\mathbb{N}}\right)$ for the disintegration of β with respect to $\Pi^{-1}\mathcal{B}_{\mathbb{C}}$, where $\mathcal{B}_{\mathbb{C}}$ is the Borel σ-algebra of $\mathbb{C}$. For the definition and basic properties of disintegrations, we refer the reader to [6, Theorem 5.14] and the discussion following it.

For $u, v \in \Lambda^*$, we denote by uv the concatenation of u with v.

Given integers $l, n \geq 1$ and $0 \leq j < l$, let $\Psi(j, l; n)$ denote the set of words $u_0 ... u_s \in \Lambda^*$ such that $u_0 \in \Lambda^j$, $u_i \in \Lambda^l$ for $1 \leq i \leq s$, $\left|\varphi'_{u_0...u_s}(0)\right| < 2^{-n}$, and $\left|\varphi'_{u_0...u_i}(0)\right| \geq 2^{-n}$ for $0 \leq i < s$. By bounded distortion, there exists a constant $C(l) > 1$ such that

$$2^{-n}/C(l) \leq |\varphi'_u(0)| < 2^{-n} \text{ for all } u \in \Psi(j, l; n). \tag{2.7}$$

Additionally, note that $\Psi(j, l; n)$ is a minimal cut-set for Λ^*, which means that for each $\omega \in \Lambda^{\mathbb{N}}$ there exists a unique $u \in \Psi(j, l; n)$ with $\omega \in [u]$. From this, and the relation $\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\mu$, it follows that

$$\mu = \sum_{u\in\Psi(j,l;n)} p_u \cdot \varphi_u\mu. \tag{2.8}$$

We shall write Ψ_n in place of $\Psi(0, 1; n)$.

It will sometimes be useful to choose words from Λ^n and $\Psi(j, l; n)$ at random. Let $\mathbf{U}_n$ and $\mathbf{I}(j, l; n)$ denote the random words with

$$\mathbb{P}\{\mathbf{U}_n = u\} = \begin{cases} p_u & \text{if } u \in \Lambda^n \\ 0 & \text{otherwise} \end{cases} \text{ and } \mathbb{P}\{\mathbf{I}(j, l; n) = u\} = \begin{cases} p_u & \text{if } u \in \Psi(j, l; n) \\ 0 & \text{otherwise} \end{cases}.$$

The following lemma will be needed in Section 4 when we establish the non-saturation of components discussed in Section 1.6. The lemma explains why Ψ_n and $\mathbf{I}(0, 1; n)$ are not sufficient, and why the more general $\Psi(j, l; n)$ and $\mathbf{I}(j, l; n)$ are required.

Lemma 2.8. *For every sufficiently large $l \geq 1$ there exists $M = M(l) \in \mathbb{Z}_{>0}$ such that for all $0 \leq j < l$ and $n \geq 1$ we have*

$$\mathbb{E}_{1\leq k\leq n}\left(\delta_{\mathbf{U}_{j+lk}}\right) \ll \mathbb{E}_{1\leq k\leq nM}\left(\delta_{\mathbf{I}(j,l;k)}\right), \tag{2.9}$$

with Radon–Nikodym derivative bounded by M.

Remark. Note that $\mathbb{E}_{1\leq k\leq n}\left(\delta_{\mathbf{U}_{j+lk}}\right)$ denotes the probability measure $\frac{1}{n}\sum_{k=1}^n \mathbb{E}\left(\delta_{\mathbf{U}_{j+lk}}\right)$. The measure $\mathbb{E}_{1\leq k\leq nM}\left(\delta_{\mathbf{I}(j,l;k)}\right)$ is defined analogously.

Proof. Let $C \geq 1$ be the constant obtained in Lemma 2.2, set

$$r_{\max} := \sup\{|\varphi'_i(z)| \,:\, i \in \Lambda \text{ and } z \in \Omega\} \text{ and } r_{\min} := \inf\{|\varphi'_i(z)| \,:\, i \in \Lambda \text{ and } z \in \Omega\},$$

let $l \geq 1$ be such that $Cr^l_{\max} < 1/2$, and let $M \geq 1$ be with $r^{2l}_{\min} \geq 2^{-M}$. For $u \in \Lambda^*$ and $v \in \Lambda^l$,

$$|\varphi'_{uv}(0)| = |\varphi'_u(\varphi_v(0))|\,|\varphi'_v(0)| \leq Cr^l_{\max}\,|\varphi'_u(0)| < |\varphi'_u(0)|/2. \tag{2.10}$$

Let $0 \leq j < l$, $n \geq 1$, $1 \leq k \leq n$, $u_0 \in \Lambda^j$, and $u_1, ..., u_k \in \Lambda^l$ be given. Let $m \geq 1$ be the unique integer with

$$2^{-m-1} \leq \left|\varphi'_{u_0...u_k}(0)\right| < 2^{-m}.$$

From

$$\left|\varphi'_{u_0...u_k}(0)\right| \geq r^{l(n+1)}_{\min} \geq 2^{-Mn}$$

it follows that $m \le Mn$. Moreover, by 2.10,

$$\left|\varphi'_{u_0\ldots u_k}(0)\right| \le \left|\varphi'_{u_0\ldots u_i}(0)\right|/2 \text{ for all } 0 \le i < k,$$

which implies that $u_0 \ldots u_k \in \Psi(j,l;m)$.

We have thus shown that

$$\cup_{k=1}^{n} \Lambda^{j+lk} \subset \cup_{k=1}^{nM} \Psi(j,l;k).$$

This clearly gives (2.9) with Radon–Nikodym derivative bounded by M, which completes the proof of the lemma. □

3. The μ-measure of real-analytic curves

3.1. **Vanishing of μ on real-analytic curves.** The following proposition is the main result of this subsection.

Proposition 3.1. *For each regular real-analytic curve $\Gamma \subset \Omega$ we have $\mu(\Gamma) = 0$.*

The proof of the proposition requires some preparation.

Lemma 3.2. *Let $E \subset \Omega$ be a Borel set with $\mu(E) > 0$. Then for each $\epsilon > 0$ there exists $u \in \Lambda^*$ such that $\varphi_u\mu(E) > 1-\epsilon$ and* $\operatorname{diam}(\varphi_u(\Omega)) \le \epsilon$.

Proof. Since $\mu(E) > 0$ and $K_\Phi = \operatorname{supp}(\mu)$, the density theorem for Radon measures (see [17, Corollary 2.14]) implies that there exists $z_0 \in K_\Phi$ such that

$$\lim_{r\downarrow 0} \frac{\mu\left(E \cap \overline{D}_r(z_0)\right)}{\mu\left(\overline{D}_r(z_0)\right)} = 1. \tag{3.1}$$

Since μ is a probability measure, the set $\{r > 0 : \mu(C_r(z_0)) > 0\}$ is at most countable. Let $C > 1$ be a large global constant, and let $m \in \mathbb{Z}_{>0}$ and $\epsilon, r, \delta \in (0,1)$ be such that $C \ll m \ll \epsilon^{-1} \ll r^{-1} \ll \delta^{-1}$ and $\mu(C_r(z_0)) = 0$. Since $\epsilon^{-1} \ll r^{-1}$ and by (3.1), we may assume that

$$\mu\left(E \cap \overline{D}_r(z_0)\right) > (1-\epsilon)\mu\left(\overline{D}_r(z_0)\right).$$

Write A for the closed annulus $\overline{D}_{r+\delta}(z_0) \setminus D_{r-\delta}(z_0)$. Since $\mu(C_r(z_0)) = 0$ and $\epsilon^{-1}, r^{-1} \ll \delta^{-1}$, we may assume that $\mu(A) < \epsilon\mu\left(\overline{D}_r(z_0)\right)$.

Setting $n := \lceil -\log\delta \rceil$, let $\mathcal{U}_1$ be the set of all $u \in \Psi_{n+m}$ such that $\varphi_u\mu\left(\overline{D}_r(z_0)\right) > 0$ and let $\mathcal{U}_2$ be the set of all $u \in \Psi_{n+m}$ such that $\varphi_u(K_\Phi) \subset \overline{D}_r(z_0)$. By Lemma 2.3, from (2.7), and since $C \ll m$, for all $u \in \Psi_{n+m}$

$$\operatorname{diam}(\varphi_u(\Omega)) \le C2^{-n-m} \le \delta.$$

From this, since $K_\Phi \subset \Omega$, and by the definitions and $\mathcal{U}_1$ and $\mathcal{U}_2$, it follows that $\varphi_u(K_\Phi) \subset A$ for each $u \in \mathcal{U}_1 \setminus \mathcal{U}_2$. Hence, from (2.8),

$$\sum_{u\in\mathcal{U}_1\setminus\mathcal{U}_2} p_u \le \sum_{u\in\Psi_{n+m}} p_u \cdot \varphi_u\mu(A) = \mu(A) \le \epsilon\mu\left(\overline{D}_r(z_0)\right),$$

which gives

$$\begin{aligned}(1-\epsilon)\mu\left(\overline{D}_r(z_0)\right) &< \mu\left(E \cap \overline{D}_r(z_0)\right)\\ &\le \sum_{u\in\mathcal{U}_1} p_u \cdot \varphi_u\mu(E) \le \epsilon\mu\left(\overline{D}_r(z_0)\right) + \sum_{u\in\mathcal{U}_2} p_u \cdot \varphi_u\mu(E).\end{aligned}$$

Thus,

$$(1-2\epsilon)\sum_{u\in\mathcal{U}_2} p_u \le (1-2\epsilon)\mu\left(\overline{D}_r(z_0)\right) < \sum_{u\in\mathcal{U}_2} p_u\cdot\varphi_u\mu\left(E\right),$$

from which it follows that there exists $u\in\mathcal{U}_2$ such that $\varphi_u\mu\left(E\right) > 1-2\epsilon$. Since $\operatorname{diam}\left(\varphi_u\left(\Omega\right)\right)\le\delta<\epsilon$, this completes the proof of the lemma. $\square$

We omit the proof of the following lemma, which follows easily from analyticity and the fact that harmonic functions are real-analytic.

Lemma 3.3. *Let $\Gamma\subset\Omega$ be a connected regular real-analytic curve, let $h:\Omega\to\mathbb{R}$ be harmonic, and suppose that the set $h^{-1}\{0\}\cap\Gamma$ has a limit point in Γ. Then $h(z)=0$ for all $z\in\Gamma$.*

Proof of Proposition 3.1. Assume by contradiction that μ gives positive mass to some regular real-analytic curve. From this assumption and by Definition 1.3, it follows easily that there exist a compact interval $J\subset\mathbb{R}$, a domain $V\subset\mathbb{C}$ with $J\subset V$, and a holomorphic injection $f:V\to\Omega$, such that $\mu\left(f(J)\right)>0$. Writing $h:=\operatorname{Im}\circ f^{-1}$, it holds that $h:f(V)\to\mathbb{R}$ is harmonic and regular (i.e. $\nabla h(z)\neq 0$ for all $z\in f(V)$, where ∇h denotes the gradient of h). Moreover, $f(J)\subset h^{-1}\{0\}$.

Let $W\subset\mathbb{C}$ be a bounded domain with $f(J)\subset W\subset\overline{W}\subset f(V)$. By Lemma 3.2, there exists a sequence $\{u_n\}_{n\ge1}\subset\Lambda^*$ such that $\varphi_{u_n}\mu\left(f(J)\right)>1-\frac{1}{n}$ and $\varphi_{u_n}(\Omega)\subset W$ for each $n\ge1$. For $n\ge1$ set $r_n:=\left|\varphi_{u_n}'(0)\right|$ and $h_n:=S_{r_n}^{-1}\circ h\circ\varphi_{u_n}$, where recall that $S_{r_n}z:=r_nz$ for $z\in\mathbb{C}$. Since h is harmonic and φ_{u_n} is holomorphic, it follows that $h_n:\Omega\to\mathbb{R}$ is harmonic.

It is clear that h is Lipschitz on $\overline{W}$. From this and by Lemma 2.3, it follows that there exists $M\ge1$ such that h_n is M-Lipschitz on Ω for each $n\ge1$. Given $n\ge1$,

$$\varphi_{u_n}\mu\left(h^{-1}\{0\}\right)\ge\varphi_{u_n}\mu\left(f(J)\right)>0,$$

and so there exists $z_n\in\Omega$ such that $h_n(z_n)=0$. Thus, for each $z\in\Omega$,

$$|h_n(z)|=|h_n(z)-h_n(z_n)|\le M\cdot\operatorname{diam}(\Omega),$$

which shows that $\{h_n\}_{n\ge1}$ is uniformly bounded on Ω. Hence, by [1, Theorems 1.23 and 2.6] and by moving to a subsequence without changing notation, there exists a harmonic function $h_0:\Omega\to\mathbb{R}$ such that $h_n\overset{n}{\to}h_0$ and $\nabla h_n\overset{n}{\to}\nabla h_0$ uniformly on compact subsets of Ω.

Since $\overline{W}$ is compact and h is regular,

$$\min\left\{|\nabla h(z)|\;:\;z\in\overline{W}\right\}>0.$$

Hence, by bounded distortion and the chain rule,

$$\inf\left\{|\nabla h_n(z)|\;:\;n\ge1\text{ and }z\in\Omega\right\}>0.$$

From this, and since $\nabla h_n\overset{n}{\to}\nabla h_0$ uniformly on compact subsets of Ω, it follows that h_0 is regular. Thus, since h_0 is real-analytic (being harmonic) and by the real-analytic implicit function theorem (see [16, Theorem 2.3.5]), we obtain that $h_0^{-1}\{0\}$ is a regular real-analytic curve.

Let us show that $\mu\left(h_0^{-1}\{0\}\right)=1$. Let $\epsilon>0$ be given. For each $n\ge1$,

$$\mu\left(h_n^{-1}\{0\}\right)=\varphi_{u_n}\mu\left(h^{-1}\{0\}\right)\ge\varphi_{u_n}\mu\left(f(J)\right)>1-\frac{1}{n}.$$

From this and since $h_n \overset{n}{\to} h_0$ uniformly on compact subsets of Ω, it follows that there exists $n \geq 1$ such that $\mu\left(h_n^{-1}\{0\}\right) > 1-\epsilon$ and $\|h_0 - h_n\|_{K_\Phi} < \epsilon$, where $\|\cdot\|_{K_\Phi}$ denotes the supremum norm on K_Φ. Given $z \in K_\Phi \cap h_n^{-1}\{0\}$,

$$|h_0(z)| = |h_0(z) - h_n(z)| < \epsilon,$$

from which it follows that

$$\mu\left(h_0^{-1}(-\epsilon,\epsilon)\right) \geq \mu\left(K_\Phi \cap h_n^{-1}\{0\}\right) > 1-\epsilon.$$

Since this holds for all $\epsilon > 0$ and $h_0^{-1}\{0\} = \cap_{\epsilon>0} h_0^{-1}(-\epsilon,\epsilon)$, we obtain that $\mu\left(h_0^{-1}\{0\}\right) = 1$.

Since $h_0^{-1}\{0\}$ is a closed subset of Ω and K_Φ is the support of μ, it follows that $K_\Phi \subset h_0^{-1}\{0\}$. Let Γ denote the union of all connected components of $h_0^{-1}\{0\}$ whose intersection with K_Φ is nonempty. Note that Γ is a regular real-analytic curve.

Since $h_0^{-1}\{0\}$ is a manifold it is locally connected, and so its connected components are open in $h_0^{-1}\{0\}$. Thus, Γ is closed in $h_0^{-1}\{0\}$, from which it follows that Γ is also closed in Ω. We shall arrive at the desired contradiction by showing that Γ is Φ-invariant.

Let Γ_0 be a connected component of $h_0^{-1}\{0\}$ such that $\Gamma_0 \subset \Gamma$, and fix $z_0 \in \Gamma_0 \cap K_\Phi$. Since Γ_0 is open in $h_0^{-1}\{0\}$, there exists $\epsilon > 0$ such that

$$D_\epsilon(z_0) \cap h_0^{-1}\{0\} \subset \Gamma_0. \tag{3.2}$$

Let $u \in \Lambda^*$ be such that $z_0 \in \varphi_u(K_\Phi)$ and $\operatorname{diam}(\varphi_u(K_\Phi)) < \epsilon$. From these properties, since $\varphi_u(K_\Phi) \subset K_\Phi \subset h_0^{-1}\{0\}$, and by (3.2), we obtain that $\varphi_u(K_\Phi) \subset \Gamma_0$.

Given $i \in \Lambda$ we have $\varphi_i(\varphi_u(K_\Phi)) \subset K_\Phi \subset h_0^{-1}\{0\}$, and so

$$\varphi_u(K_\Phi) \subset (h_0 \circ \varphi_i)^{-1}\{0\} \cap \Gamma_0. \tag{3.3}$$

Note that, since the maps in Φ do not have a common fixed point, the set K_Φ is necessarily uncountable, and so $\varphi_u(K_\Phi)$ is also uncountable. From this and (3.3), it follows that $(h_0 \circ \varphi_i)^{-1}\{0\} \cap \Gamma_0$ has a limit point in Γ_0. Thus, since $h_0 \circ \varphi_i$ is harmonic and by Lemma 3.3, $h_0(\varphi_i(z)) = 0$ for all $z \in \Gamma_0$, which implies that $\varphi_i(\Gamma_0) \subset h_0^{-1}\{0\}$.

Since $\varphi_i(\Gamma_0)$ is connected, $\varphi_i(\Gamma_0) \subset \Gamma_1$ for some connected component Γ_1 of $h_0^{-1}\{0\}$. Since $\varphi_i(\varphi_u(K_\Phi)) \subset \varphi_i(\Gamma_0) \subset \Gamma_1$ and $\varphi_i(\varphi_u(K_\Phi)) \subset K_\Phi$, we obtain that $\Gamma_1 \subset \Gamma$, and so $\varphi_i(\Gamma_0) \subset \Gamma$. We have thus shown that Γ is a Φ-invariant regular real-analytic curve which is closed in Ω. This contradicts our standing assumptions and completes the proof of the proposition. $\square$

3.2. **The μ-measure of neighborhoods of dyadic cubes.** The purpose of this subsection is to prove the following proposition. Recall that for a nonempty set $E \subset \mathbb{C}$ and $r > 0$, we write $E^{(r)}$ for the r-thickening of E. Also recall the notation $\mathcal{F}_\epsilon$ introduced in Section 2.3.

Proposition 3.4. *For each $\epsilon > 0$ there exists $\delta > 0$ such that,*

$$\psi\mu\left(\cup_{D\in\mathcal{D}_n^{\mathbb{C}}}(\partial D)^{(\delta 2^{-n})}\right) < \epsilon \text{ for all } \psi \in \mathcal{F}_\epsilon \text{ and } n \geq 1,$$

where ∂D denotes the boundary of D.

We shall need the following lemma. Its proof can be found, for example, in [28, Chapter 8, Proposition 3.5].

Lemma 3.5. *Let $\Omega_1 \subset \mathbb{C}$ be a domain, and let $f, f_1, f_2, ... : \Omega_1 \to \mathbb{C}$ be holomorphic. Suppose that $f_n \overset{n}{\to} f$ uniformly on compact subsets of Ω_1, and that f_n is injective for each $n \geq 1$. Then f is either injective or constant.*

Most of the proof of Proposition 3.4 is contained in the proof of the following lemma.

Lemma 3.6. *For each $\epsilon > 0$ there exists $\delta > 0$ such that $\psi\varphi_u\mu\left((z+\ell)^{(\delta 2^{-n})}\right) < \epsilon$ for all $n \geq 1$, $u \in \Psi_n$, $\psi \in \mathcal{F}_\epsilon$, $\ell \in \mathbb{RP}^1$ and $z \in \mathbb{C}$.*

Proof. Let $C > 1$ be a large global constant, and let $\epsilon > 0$ be given. Assume by contradiction that for every $k \geq 1$ there exists $n_k \geq 1$, $u_k \in \Psi_{n_k}$, $\psi_k \in \mathcal{F}_\epsilon$, $\ell_k \in \mathbb{RP}^1$ and $z_k \in \mathbb{C}$ such that

$$\psi_k\varphi_{u_k}\mu\left((z_k+\ell_k)^{(C^{-1}k^{-1}2^{-n_k})}\right) \geq \epsilon. \tag{3.4}$$

For $k \geq 1$ set $r_k := \left|\varphi'_{u_k}(0)\right|$, $w_k := S_{r_k}^{-1} \circ \psi_k \circ \varphi_{u_k}(0)$, and $f_k := T_{w_k}^{-1} S_{r_k}^{-1} \psi_k \varphi_{u_k}$. By (2.7) and (3.4),

$$f_k\mu\left(\left(\pi_{\ell_k^\perp}\left(r_k^{-1}z_k - w_k\right) + \ell_k\right)^{(1/k)}\right) \geq \epsilon. \tag{3.5}$$

Note that $f_k(0) = 0$. Thus, by Lemma 2.3 and since Ω is bounded, we may assume that

$$|f_k(z)| \leq C/\epsilon \text{ for all } z \in \Omega. \tag{3.6}$$

From this and by (3.5),

$$\left|\pi_{\ell_k^\perp}\left(r_k^{-1}z_k - w_k\right)\right| \leq 1 + C/\epsilon. \tag{3.7}$$

From (3.6) and (3.7), by Montel's theorem, and by moving to a subsequence without changing the notation, there exist $w \in \mathbb{C}$, $\ell \in \mathbb{RP}^1$, and a holomorphic $f : \Omega \to \mathbb{C}$ such that $\pi_{\ell_k^\perp}\left(r_k^{-1}z_k - w_k\right) \overset{k}{\to} w$, $d(\ell, \ell_k) \overset{k}{\to} 0$, and $f_k \overset{k}{\to} f$ uniformly on compact subsets of Ω. From this and by (3.5) it follows easily that $f\mu(w+\ell) \geq \epsilon$.

It also holds that $f'_k(0) \overset{k}{\to} f'(0)$ (see [28, page 54]). Thus, since $|f'_k(0)| \geq \epsilon$ for each $k \geq 1$, it follows that $|f'(0)| \geq \epsilon$. In particular, f is nonconstant. Hence, since the maps f_k are injective, Lemma 3.5 implies that f is injective as well. Since f is also holomorphic, it follows that $f^{-1}(w+\ell)$ is a regular real-analytic curve. But $\mu\left(f^{-1}(w+\ell)\right) \geq \epsilon$, which contradicts Proposition 3.1 and completes the proof of the lemma. □

Proof of Proposition 3.4. Let $0 < \epsilon, \eta, \delta < 1$ be with $\epsilon^{-1} \ll \eta^{-1} \ll \delta^{-1}$, let $\psi \in \mathcal{F}_\epsilon$ and $n \geq 1$ be given, and set

$$B := \cup_{D \in \mathcal{D}_n^{\mathbb{C}}} (\partial D)^{(\delta 2^{-n})}.$$

For $u \in \Psi_n$, it follows from Lemma 2.3 and (2.7) that

$$\operatorname{diam}\left(\operatorname{supp}\left(\psi\varphi_u\mu\right)\right) = O\left(\epsilon^{-1}2^{-n}\right).$$

Hence,

$$\#\left\{D \in \mathcal{D}_n^{\mathbb{C}} \,:\, \psi\varphi_u\mu\left((\partial D)^{(\delta 2^{-n})}\right) > 0\right\} = O_\epsilon(1).$$

Moreover, by Lemma 3.6 and since $\epsilon^{-1}, \eta^{-1} \ll \delta^{-1}$,

$$\psi\varphi_u\mu\left((\partial D)^{(\delta 2^{-n})}\right) < \eta \text{ for all } D \in \mathcal{D}_n^{\mathbb{C}}.$$

Combining these facts, we obtain

$$\psi\varphi_u\mu\,(B) = O_\epsilon(\eta) \text{ for all } u \in \Psi_n.$$

From this, together with $\psi\mu = \sum_{u\in\Psi_n} p_u \cdot \psi\varphi_u\mu$ and $\epsilon^{-1} \ll \eta^{-1}$, we get that $\psi\mu(B) < \epsilon$, which completes the proof of the proposition. ☐

4. Non-saturation of components

The following proposition is the main result of this section. Its proof is a modification of the proof of [21, Proposition 1.5], but it contains significant differences and requires several adaptations to our holomorphic setting.

Proposition 4.1. *Suppose that* $\dim\mu < 2$*. Then there exists* $\gamma > 0$ *such that for every* $\epsilon > 0$*,* $m \ge M(\epsilon) \ge 1$*,* $n \ge 1$*, and* $\psi \in \mathcal{F}_\epsilon$*, if we set* $\theta := \psi\mu$*, then*

$$\mathbb{P}\left\{\inf_{w\mathbb{R}\in\mathbb{RP}^1} \frac{1}{m} H\left(\pi_{w\mathbb{R}}\theta_{z,n}, \mathcal{D}_{n+m}\right) > \dim\mu - 1 + \gamma\right\} > 1-\epsilon.$$

Remark. Let θ be as in the statement of the proposition. Since θ has uniform entropy dimension (see Proposition 5.4 below), Proposition 4.1 implies that, in the terminology introduced by Hochman (see [11, Definition 2.3]), all but an arbitrarily small proportion of the components of θ are non-saturated in any direction.

Throughout this section, given $u \in \Lambda^*$ we shall write S_u in place of $S^{-1}_{|\varphi_u'(0)|}$. That is, $S_u z = |\varphi_u'(0)|^{-1} z$ for $z \in \mathbb{C}$. Most of the proof of Proposition 4.1 is devoted to establishing the following statement.

Proposition 4.2. *Suppose that* $\dim\mu < 2$*. Then there exists* $0 < \gamma < 1$ *such that for every* $\epsilon > 0$*,* $n \ge N(\epsilon) \ge 1$*,* $z\mathbb{R} \in \mathbb{RP}^1$*,* $u \in \Lambda^*$*, and* $\psi \in \mathcal{F}_\epsilon$*,*

$$\frac{1}{n} H\left(S_u\pi_{z\mathbb{R}}\psi\varphi_u\mu, \mathcal{D}_n\right) \ge \dim\mu - 1 + \gamma.$$

The proof of Proposition 4.2 involves bounding from below entropies of the form

$$\frac{1}{m} H\left(S_{uv}\pi_{z\mathbb{R}}\psi\varphi_{uv}\mu, \mathcal{D}_m\right), \tag{4.1}$$

where $m \ge 1$ is large, $u \in \Lambda^*$ is fixed, and $v \in \cup_{j=1}^n \Lambda^j$, with $n \ge 1$ large. Appropriate lower bounds for these entropies, combined with a multiscale decomposition, the concavity of entropy, and the stationarity of μ, will yield the proposition.

In Section 4.1, we show that the entropies in (4.1) are bounded from below by $\dim\mu - 1$ up to an arbitrarily small error. Section 4.2 is devoted to the study of the direction cocycle $\alpha_n : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ (defined in that section). We prove that it is not a coboundary, which yields an important non-concentration corollary (Corollary 4.10). In Section 4.3, we use this corollary in order to show that, when v is chosen randomly according to $\mathbb{E}_{1\le i\le n}\left(\delta_{\mathbf{U}_i}\right)$, the entropies in (4.1) are, with non-negligible probability, bounded from below by $\frac{1}{2}\dim\mu$ up to an arbitrarily small error. Note that since $\dim\mu < 2$, we have $\frac{1}{2}\dim\mu > \dim\mu - 1$. Finally, in Section 4.4, we complete the proofs of Propositions 4.1 and 4.2.

4.1. **The trivial lower bound.** The following lemma will be used several times below.

Lemma 4.3. *For every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$, $u \in \Lambda^*$, $\psi \in \mathcal{F}_\epsilon$, and $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$ with $d(z\mathbb{R}, w\mathbb{R}) \geq \epsilon$, we have*

$$\left| \frac{1}{m} H\left(S_u \psi \varphi_u \mu, \pi_{z\mathbb{R}}^{-1} \mathcal{D}_m \vee \pi_{w\mathbb{R}}^{-1} \mathcal{D}_m\right) - \dim \mu \right| < \epsilon.$$

Proof. Let $0 < \epsilon < 1$ and $m \geq 1$ be with $\epsilon^{-1} \ll m$, let $u \in \Lambda^*$ and $\psi \in \mathcal{F}_\epsilon$ be given, let $z\mathbb{R}, w\mathbb{R} \in \mathbb{RP}^1$ be with $d(z\mathbb{R}, w\mathbb{R}) \geq \epsilon$, and set

$$\mathcal{E} := \pi_{z\mathbb{R}}^{-1} \mathcal{D}_m \vee \pi_{w\mathbb{R}}^{-1} \mathcal{D}_m.$$

From $d(z\mathbb{R}, w\mathbb{R}) \geq \epsilon$ it follows easily that the partitions $\mathcal{E}$ and $\mathcal{D}_m^{\mathbb{C}}$ are $O_\epsilon(1)$-commensurable. That is, for each $E \in \mathcal{E}$ and $D \in \mathcal{D}_m^{\mathbb{C}}$

$$\#\left\{D' \in \mathcal{D}_m^{\mathbb{C}} \;:\; D' \cap E \neq \emptyset\right\}, \#\left\{E' \in \mathcal{E} \;:\; E' \cap D \neq \emptyset\right\} = O_\epsilon(1).$$

Hence, by [9, Lemma 3.2] and since $\epsilon^{-1} \ll m$,

$$\left| \frac{1}{m} H\left(S_u \psi \varphi_u \mu, \mathcal{E}\right) - \frac{1}{m} H\left(S_u \psi \varphi_u \mu, \mathcal{D}_m\right) \right| < \epsilon.$$

By Lemma 2.3, $S_u \circ \psi \circ \varphi_u$ is bi-Lipschitz with bi-Lipschitz constant $O(1/\epsilon)$. Thus, by Lemma 2.6,

$$\left| \frac{1}{m} H\left(S_u \psi \varphi_u \mu, \mathcal{D}_m\right) - \frac{1}{m} H\left(\mu, \mathcal{D}_m\right) \right| < \epsilon.$$

Moreover, by (2.5),

$$\left| \frac{1}{m} H\left(\mu, \mathcal{D}_m\right) - \dim \mu \right| < \epsilon.$$

All of this completes the proof of the lemma. $\square$

We can now obtain the following lower bound.

Lemma 4.4. *For every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, $u \in \Lambda^*$, and $\psi \in \mathcal{F}_\epsilon$,*

$$\frac{1}{m} H\left(S_u \pi_{z\mathbb{R}} \psi \varphi_u \mu, \mathcal{D}_m\right) > \dim \mu - 1 - \epsilon.$$

Proof. Let $0 < \epsilon < 1$ and $m \geq 1$ be with $\epsilon^{-1} \ll m$, let $z\mathbb{R} \in \mathbb{RP}^1$, $u \in \Lambda^*$, and $\psi \in \mathcal{F}_\epsilon$ be given, and set $\theta := S_u \psi \varphi_u \mu$. Recall that $(z\mathbb{R})^\perp \in \mathbb{RP}^1$ denotes the line perpendicular to $z\mathbb{R}$, and set

$$\mathcal{E} := \pi_{z\mathbb{R}}^{-1} \mathcal{D}_m \vee \pi_{(z\mathbb{R})^\perp}^{-1} \mathcal{D}_m.$$

By Lemma 4.3,

$$\left| \frac{1}{m} H\left(\theta, \mathcal{E}\right) - \dim \mu \right| < \epsilon.$$

By Lemma 2.3,

$$\operatorname{diam}\left(\operatorname{supp}\left(\theta\right)\right) = O\left(1/\epsilon\right).$$

Hence, since $\epsilon^{-1} \ll m$,

$$\frac{1}{m} H\left(\pi_{(z\mathbb{R})^\perp} \theta, \mathcal{D}_m\right) \leq 1 + \epsilon.$$

Additionally, by the conditional entropy formula,

$$\frac{1}{m}H\left(\theta,\mathcal{E}\right)\leq\frac{1}{m}H\left(\pi_{z\mathbb{R}}\theta,\mathcal{D}_{m}\right)+\frac{1}{m}H\left(\pi_{(z\mathbb{R})^{\perp}}\theta,\mathcal{D}_{m}\right).$$

All of this gives

$$\frac{1}{m}H\left(\pi_{z\mathbb{R}}\theta,\mathcal{D}_{m}\right)\geq\dim\mu-1-2\epsilon.$$

Since $S_u\circ\pi_{z\mathbb{R}}=\pi_{z\mathbb{R}}\circ S_u$, this completes the proof of the lemma. □

4.2. **The direction cocycle.** Let $\alpha:\Lambda^{\mathbb{N}}\to\mathbb{RP}^1$ be such that

$$\alpha(\omega)=\varphi_{\omega_0}'\left(\Pi(\sigma\omega)\right)\mathbb{R}\text{ for }\omega\in\Lambda^{\mathbb{N}}.$$

Define a cocycle, which we call the direction cocycle, by setting for each $n\geq 0$ and $\omega\in\Lambda^{\mathbb{N}}$

$$\alpha_n(\omega):=\prod_{i=0}^{n-1}\alpha\left(\sigma^i\omega\right),$$

where recall from Section 2.1 that $\mathbb{RP}^1$ is considered as a multiplicative group. Note that $\alpha_0(\omega)=\mathbb{R}$, and that by the chain rule

$$\alpha_n(\omega)=\varphi_{\omega|_n}'\left(\Pi\left(\sigma^n\omega\right)\right)\mathbb{R}. \tag{4.2}$$

The following proposition is the main result of this subsection.

Proposition 4.5. *There does not exist a Borel measurable map $f:\Lambda^{\mathbb{N}}\to\mathbb{RP}^1$ such that $\alpha(\omega)=f(\omega)^{-1}f\left(\sigma\omega\right)$ for β-a.e. ω.*

Remark. In the terminology of measurable cohomology (see [24]), Proposition 4.5 asserts that the cocycle α_n is not a coboundary.

The proof of the proposition requires some preparation. We omit the proof of the following lemma, which follows easily from the real-analytic implicit function theorem (see [16, Theorem 2.3.5]) and the real-analyticity of harmonic functions.

Lemma 4.6. *Let $\Omega_1\subset\mathbb{C}$ be a domain, and let $h:\Omega_1\to\mathbb{R}$ be a nonconstant harmonic function. Then for each $c\in\mathbb{R}$ there exists a (possibly empty) countable set $A\subset\Omega_1$ and a (possibly empty) regular real-analytic curve $\Gamma\subset\Omega_1$ such that $h^{-1}\{c\}=A\cup\Gamma$.*

Using this lemma, we can establish the following statement.

Lemma 4.7. *Let $\psi:\Omega\to\mathbb{C}$ be holomorphic and injective, and let $j\in\Lambda$ be given. Set $g:=\psi\circ\varphi_j\circ\psi^{-1}$, and suppose that*

$$g'\left(\psi(z)\right)\mathbb{R}=\mathbb{R}\text{ for all }z\in K_\Phi. \tag{4.3}$$

Then g is homothetic.

Proof. Set $U:=\mathbb{C}\setminus i\mathbb{R}_{\geq 0}$ and

$$\Omega_1:=\left\{z\in\Omega\,:\,g'\left(\psi(z)\right)\in U\right\},$$

so that Ω_1 is an open subset of Ω. From (4.3) and since $\mathrm{supp}(\mu)=K_\Phi$, it follows that $\mu(\Omega_1)=1$. Let $\log_U$ denote the natural branch of the logarithm corresponding to the domain U, so that

$$\mathrm{Im}\left(\log_U(z)\right)\in\left(-\frac{3\pi}{2},\frac{\pi}{2}\right)\text{ for all }z\in U.$$

Setting

$$h := \operatorname{Im} \circ \log_U \circ g' \circ \psi,$$

it holds that $h : \Omega_1 \to \mathbb{R}$ is harmonic.

Let Ω_2 be a connected component of Ω_1 with $\mu(\Omega_2) > 0$. Assume by contradiction that h is nonconstant on Ω_2, and set

$$L := \Omega_2 \cap h^{-1}\{0, -\pi\}.$$

By Lemma 4.6, there exists a (possibly empty) countable set $A \subset \Omega_2$ and a (possibly empty) regular real-analytic curve $\Gamma \subset \Omega_2$ such that $L = A \cup \Gamma$. On the other hand, by (4.3) we have $\mu\left(h^{-1}\{0, -\pi\}\right) = 1$, and so $\mu(L) = \mu(\Omega_2) > 0$. Since the maps in Φ do not have a common fixed point, μ is clearly nonatomic. Hence $\mu(A) = 0$, which implies that $\mu(\Gamma) = \mu(L) > 0$. But this contradicts Proposition 3.1, and so h must be constant on Ω_2.

Since h is constant on Ω_2, there exists $0 \neq z_0 \in \mathbb{C}$ such that $g'(\psi(\Omega_2)) \subset z_0\mathbb{R}_{>0}$. Thus, since $\psi(\Omega_2)$ is open and $z_0\mathbb{R}_{>0}$ has empty interior, the open mapping theorem implies that g' is constant on $\psi(\Omega)$. From this and (4.3), it follows that there exists $0 \neq r \in \mathbb{R}$ such that $g'(z) = r$ for all $z \in \psi(\Omega)$, which shows that g is homothetic. $\square$

Proof of Proposition 4.5. Assume by contradiction that there exists a Borel measurable $f : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ such that $\alpha(\omega) = f(\omega)^{-1} f(\sigma\omega)$ for β-a.e. ω. Then by (4.2),

$$\varphi'_{\omega|_n}(\Pi(\sigma^n\omega))\mathbb{R} = f(\omega)^{-1} f(\sigma^n\omega) \text{ for all } n \geq 0 \text{ and } \beta\text{-a.e. } \omega,$$

which implies that

$$f(\omega) = \varphi'_u(\Pi(\omega))\mathbb{R} f(u\omega) \text{ for all } u \in \Lambda^* \text{ and } \beta\text{-a.e. } \omega. \tag{4.4}$$

By Lusin's theorem, there exists a Borel subset E of $\Lambda^{\mathbb{N}}$ such that $\beta(E) > 0$ and $f|_E$ is continuous, where $f|_E$ denotes the restriction of f to E. By the martingale theorem, there exists $\eta \in E$ such that $\lim_{n\to\infty} \beta_{[\eta|_n]}(E) = 1$. Note that, for each $n \geq 1$, the measure $\beta_{[\eta|_n]}$ equals the pushforward of β via the map $\omega \mapsto \eta|_n\omega$. Thus,

$$\lim_{n\to\infty} \beta\{\omega : \eta|_n\omega \in E\} = 1.$$

Given $\epsilon > 0$, this implies that

$$\lim_{n\to\infty} \beta\{\omega : d(f(\eta|_n\omega), f(\eta)) > \epsilon\} = 0.$$

In other words, the sequence $\{\omega \mapsto f(\eta|_n\omega)\}_{n\geq 1}$ converges in probability to $f(\eta)$. Hence, there exists a strictly increasing sequence $(n_k)_{k\geq 1} \subset \mathbb{Z}_{>0}$ such that

$$\lim_{k\to\infty} f(\eta|_{n_k}\omega) = f(\eta) \text{ for } \beta\text{-a.e } \omega. \tag{4.5}$$

For $k \geq 1$, write

$$\psi_k := S_{\eta|_{n_k}} \circ T^{-1}_{\varphi_{\eta|_{n_k}}(0)} \circ \varphi_{\eta|_{n_k}}.$$

By Lemma 2.3 and since $\psi_k(0) = 0$ for $k \geq 1$, the sequence $\{\psi_k\}_{k\geq 1}$ is uniformly bounded on Ω. Thus, by Montel's theorem and by moving to a subsequence without changing notation, there exists a holomorphic $\psi : \Omega \to \mathbb{C}$ such that $\psi_k \xrightarrow{k} \psi$ uniformly on compact subsets of Ω.

It also holds that $\psi_k' \xrightarrow{k} \psi'$ uniformly on compact subsets of Ω (see [28, page 54]). Moreover, by bounded distortion,

$$\inf\left\{|\psi_k'(z)| \ : \ k \geq 1 \text{ and } z \in \Omega\right\} > 0,$$

and so $\psi'(z) \neq 0$ for all $z \in \Omega$. Hence, for each $z \in \Omega$

$$\lim_{k\to\infty} \varphi_{\eta|_{n_k}}'(z)\mathbb{R} = \lim_{k\to\infty} \psi_k'(z)\mathbb{R} = \psi'(z)\mathbb{R}.$$

From this, (4.4), and (4.5), it follows that for β-a.e. ω

$$f(\omega) = \lim_{k\to\infty} \varphi_{\eta|_{n_k}}'\left(\Pi\left(\omega\right)\right)\mathbb{R}f\left(\eta|_{n_k}\omega\right) = \psi'\left(\Pi\left(\omega\right)\right)\mathbb{R}f(\eta). \tag{4.6}$$

Recall that the maps in Φ are all assumed to be injective on Ω, from which it follows that ψ_k is injective on Ω for each $k \geq 1$. Moreover, since ψ has nonvanishing derivative it is certainly nonconstant. Hence, by Lemma 3.5, we obtain that ψ is injective.

Fix $j \in \Lambda$ and set $g := \psi \circ \varphi_j \circ \psi^{-1}$. By (4.4)

$$f(j\omega)\varphi_j'\left(\Pi\left(\omega\right)\right)\mathbb{R}f(\omega)^{-1} = \mathbb{R} \text{ for } \beta\text{-a.e. } \omega.$$

Thus, by (4.6)

$$\psi'\left(\Pi\left(j\omega\right)\right)\varphi_j'\left(\Pi\left(\omega\right)\right)\psi'\left(\Pi\left(\omega\right)\right)^{-1}\mathbb{R} = \mathbb{R} \text{ for } \beta\text{-a.e. } \omega.$$

Note that $\Pi\left(j\omega\right) = \varphi_j\left(\Pi(\omega)\right)$ for $\omega \in \Lambda^{\mathbb{N}}$, and $\psi'(z)^{-1} = \left(\psi^{-1}\right)'\left(\psi(z)\right)$ for $z \in \Omega$. Hence, for β-a.e. ω

$$g'\left(\psi \circ \Pi(\omega)\right)\mathbb{R} = \psi'\left(\varphi_j\left(\Pi(\omega)\right)\right)\varphi_j'\left(\Pi\left(\omega\right)\right)\left(\psi^{-1}\right)'\left(\psi \circ \Pi(\omega)\right)\mathbb{R} = \mathbb{R}. \tag{4.7}$$

Since $\mu = \Pi\beta$ and $\text{supp}(\mu) = K_\Phi$, this clearly implies that $g'\left(\psi(z)\right)\mathbb{R} = \mathbb{R}$ for all $z \in K_\Phi$. Thus, by Lemma 4.7, g is homothetic.

We have thus shown that for each $j \in \Lambda$ the map $\psi \circ \varphi_j \circ \psi^{-1} : \psi(\Omega) \to \mathbb{C}$ is homothetic, and so Φ is holomorphically conjugate to a homothetic IFS, which contradicts our standing assumption. Hence, there does not exist a Borel measurable $f : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ such that $\alpha(\omega) = f(\omega)^{-1}f\left(\sigma\omega\right)$ for β-a.e. ω, which completes the proof of the proposition. □

We next state an important implication of Proposition 4.5. First, we need the following definition.

Definition 4.8. Given $\delta > 0$, we say that $\theta \in \mathcal{M}\left(\mathbb{RP}^1\right)$ is δ-concentrated if there exists $z\mathbb{R} \in \mathbb{RP}^1$ such that $\theta\left(B\left(z\mathbb{R}, \delta\right)\right) > 1 - \delta$.

The following proposition is a consequence of Proposition 4.5. The derivation is identical to that of [21, Proposition 4.8] from [21, Proposition 4.6] and is therefore omitted.

Proposition 4.9. *There exists $\delta > 0$ such that for every continuous $h : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ and for β-a.e. ω, the sequence $\left(\alpha_n(\omega)h\left(\sigma^n\omega\right)\right)_{n\geq 0}$ is equidistributed with respect to some $\theta \in \mathcal{M}\left(\mathbb{RP}^1\right)$ that is not δ-concentrated.*

The following corollary follows immediately from Proposition 4.9. Recall that λ_n denotes the uniform probability measure on $\mathcal{N}_n := \{1, ..., n\}$.

Corollary 4.10. *There exists $0 < \delta < 1$ such that for every continuous $h : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ and for β-a.e. ω, there exists $N_{h,\omega} \geq 1$ so that for every $n \geq N_{h,\omega}$,*

$$\lambda_n\left\{i \in \mathcal{N}_n \ : \ d\left(\alpha_i(\omega)h\left(\sigma^i\omega\right), z\mathbb{R}\right) > \delta\right\} > \delta \text{ for all } z\mathbb{R} \in \mathbb{RP}^1.$$

4.3. **The nontrivial lower bound.** The purpose of this subsection is to prove the following proposition. Recall the random words $\mathbf{U}_i$ introduced in Section 2.8.

Proposition 4.11. *There exists* $0<\delta<1$ *such that for every* $\epsilon>0$, $m\geq M(\epsilon)\geq 1$, $n\geq N(\epsilon,m)\geq 1$, $z\mathbb{R}\in\mathbb{RP}^1$, $u\in\Lambda^*$, *and* $\psi\in\mathcal{F}_\epsilon$,

$$\mathbb{P}_{1\leq i\leq n}\left\{\frac{1}{m}H\left(S_{u\mathbf{U}_i}\pi_{z\mathbb{R}}\psi\varphi_{u\mathbf{U}_i}\mu,\mathcal{D}_m\right)>\frac{1}{2}\dim\mu-\epsilon\right\}>\delta.$$

The proof of the proposition requires some preparation.

Lemma 4.12. *For every* $\epsilon>0$, $m\geq M(\epsilon)\geq 1$, *and* $u\in\Lambda^*$, *there exists* $z\mathbb{R}\in\mathbb{RP}^1$ *such that*

$$\frac{1}{m}H\left(S_u\pi_{w\mathbb{R}}\varphi_u\mu,\mathcal{D}_m\right)>\frac{1}{2}\dim\mu-\epsilon \text{ for all } w\mathbb{R}\in\mathbb{RP}^1\setminus B\left(z\mathbb{R},\epsilon\right).$$

Proof. Let $0<\epsilon<1$ and $m\geq 1$ be with $\epsilon^{-1}\ll m$, and let $u\in\Lambda^*$ be given. For each $w\mathbb{R}\in\mathbb{RP}^1$ set

$$H\left(w\mathbb{R}\right):=\frac{1}{m}H\left(S_u\pi_{w\mathbb{R}}\varphi_u\mu,\mathcal{D}_m\right),$$

and let $z\mathbb{R}\in\mathbb{RP}^1$ be such that

$$H\left(z\mathbb{R}\right)\leq\inf_{w\mathbb{R}\in\mathbb{RP}^1}H\left(w\mathbb{R}\right)+\epsilon.$$

From Lemma 4.3, by the conditional entropy formula, and by the last inequality, it follows that for each $w\mathbb{R}\in\mathbb{RP}^1\setminus B\left(z\mathbb{R},\epsilon\right)$

$$\dim\mu-\epsilon<H\left(z\mathbb{R}\right)+H\left(w\mathbb{R}\right)\leq 2H\left(w\mathbb{R}\right)+\epsilon,$$

which proves the lemma. $\square$

The proof of Proposition 4.11 relies on a linearization argument which is contained in the proof of the following lemma.

Lemma 4.13. *For every* $\epsilon>0$, $m\geq M(\epsilon)\geq 1$, $k\geq K(\epsilon,m)\geq 1$, $u\in\Lambda^*$, $\psi\in\mathcal{F}_\epsilon$, $i\in\mathbb{Z}_{>0}$, $\omega\in\Lambda^{\mathbb{N}}$, *and* $z\mathbb{R}\in\mathbb{RP}^1$, *we have*

$$\frac{1}{m}H\left(S_{u\omega|_{i+k}}\pi_{z\mathbb{R}}\psi\varphi_{u\omega|_{i+k}}\mu,\mathcal{D}_m\right)\geq\frac{1}{m}H\left(S_{(\sigma^i\omega)|_k}\pi_\ell\varphi_{(\sigma^i\omega)|_k}\mu,\mathcal{D}_m\right)-\epsilon,$$

where

$$\ell:=\frac{z}{\left(\psi\circ\varphi_u\right)'\left(\Pi\omega\right)}\mathbb{R}\alpha_i(\omega)^{-1}\in\mathbb{RP}^1. \tag{4.8}$$

Proof. Let $C>1$ be a large global constant, and let $\epsilon\in(0,1)$ and $m,k\in\mathbb{Z}_{>0}$ be such that $C,\epsilon^{-1}\ll m\ll k$. Let $u\in\Lambda^*$, $\psi\in\mathcal{F}_\epsilon$, $i\in\mathbb{Z}_{>0}$, $\omega\in\Lambda^{\mathbb{N}}$, and $z\mathbb{R}\in\mathbb{RP}^1$ be given, and set

$$H:=\frac{1}{m}H\left(S_{u\omega|_{i+k}}\pi_{z\mathbb{R}}\psi\varphi_{u\omega|_{i+k}}\mu,\mathcal{D}_m\right).$$

Write

$$r_{\max}:=\sup\left\{\left|\varphi_j'(w)\right| : j\in\Lambda \text{ and } w\in\Omega\right\}.$$

By Lemma 2.3,

$$\operatorname{diam}\left(\varphi_{(\sigma^i\omega)|_k}(\Omega)\right)\leq C\left|\varphi'_{(\sigma^i\omega)|_k}(0)\right|\leq Cr_{\max}^k.$$

Hence, setting $a := \Pi\left(\sigma^i\omega\right)$ and $q := P_{1,a}\left(\psi\circ\varphi_{u\omega|_i}\right)$, it follows from Lemma 2.5 and since $a \in \varphi_{(\sigma^i\omega)|_k}(\Omega)$ that for all $w \in \varphi_{(\sigma^i\omega)|_k}(\Omega)$

$$\begin{aligned}\left|\psi\circ\varphi_{u\omega|_i}(w) - q(w)\right| &\leq C^3\epsilon^{-1}\left|\varphi'_{u\omega|_i}(0)\right|\left|\varphi'_{(\sigma^i\omega)|_k}(0)\right|^2 \\ &\leq C^3\epsilon^{-1}r_{\max}^k\left|\varphi'_{u\omega|_i}(0)\right|\left|\varphi'_{(\sigma^i\omega)|_k}(0)\right|.\end{aligned}$$

From this, by bounded distortion, by Lemma 2.7, and since $C, \epsilon^{-1} \ll m \ll k$,

$$H + \epsilon \geq \frac{1}{m}H\left(S_{u\omega|_{i+k}}\pi_{z\mathbb{R}}q\varphi_{(\sigma^i\omega)|_k}\mu, \mathcal{D}_m\right).$$

Set $b := \frac{\left(\psi\circ\varphi_{u\omega|_i}\right)'(a)}{\left|\left(\psi\circ\varphi_{u\omega|_i}\right)'(a)\right|}$, and recall that $S_b w = bw$ for $w \in \mathbb{C}$. Since S_b^{-1} is an isometry of $\mathbb{C}$,

$$H + 2\epsilon \geq \frac{1}{m}H\left(S_{u\omega|_{i+k}}S_b^{-1}\pi_{z\mathbb{R}}q\varphi_{(\sigma^i\omega)|_k}\mu, \mathcal{D}_m\right). \tag{4.9}$$

By the definition of the orthogonal projections and from (4.2),

$$S_b^{-1}\circ\pi_{z\mathbb{R}}\circ S_b = \pi_{b^{-1}z\mathbb{R}} = \pi_\ell,$$

where $\ell \in \mathbb{RP}^1$ is defined in (4.8). Thus, for $w \in \mathbb{C}$,

$$\begin{aligned}S_b^{-1}\circ\pi_{z\mathbb{R}}\circ q(w) &= S_b^{-1}\circ\pi_{z\mathbb{R}}\circ q(0) + S_b^{-1}\circ\pi_{z\mathbb{R}}\left(\left(\psi\circ\varphi_{u\omega|_i}\right)'(a)\cdot w\right) \\ &= S_b^{-1}\circ\pi_{z\mathbb{R}}\circ q(0) + \left|\left(\psi\circ\varphi_{u\omega|_i}\right)'(a)\right|\cdot\pi_\ell(w).\end{aligned}$$

From this, from (4.9), by bounded distortion, since $\psi \in \mathcal{F}_\epsilon$, and since $\epsilon^{-1} \ll m$,

$$H + 3\epsilon \geq \frac{1}{m}H\left(S_{(\sigma^i\omega)|_k}\pi_\ell\varphi_{(\sigma^i\omega)|_k}\mu, \mathcal{D}_m\right),$$

which completes the proof of the lemma. □

Proof of Proposition 4.11. Let $0 < \delta < 1$ be as obtained in Corollary 4.10, let $0 < \epsilon < \delta/3$ and $m, k, n \in \mathbb{Z}_{>0}$ be with $\epsilon^{-1} \ll m \ll k \ll n$, and fix $z\mathbb{R} \in \mathbb{RP}^1$, $u \in \Lambda^*$, and $\psi \in \mathcal{F}_\epsilon$. Set

$$H(v) := \frac{1}{m}H\left(S_{uv}\pi_{z\mathbb{R}}\psi\varphi_{uv}\mu, \mathcal{D}_m\right) \text{ for } v \in \Lambda^*,$$

and let

$$P := \mathbb{P}_{1\leq i\leq n}\left\{H\left(\mathbf{U}_i\right) > \frac{1}{2}\dim\mu - \epsilon\right\}.$$

Recalling that λ_n denotes the uniform probability measure on $\mathcal{N}_n := \{1, ..., n\}$,

$$\begin{aligned}P &= \int\beta\left\{\omega : H\left(\omega|_i\right) > \frac{1}{2}\dim\mu - \epsilon\right\}d\lambda_n(i) \\ &= \int\lambda_n\left\{i \in \mathcal{N}_n : H\left(\omega|_i\right) > \frac{1}{2}\dim\mu - \epsilon\right\}d\beta(\omega).\end{aligned}$$

Hence, since $\epsilon^{-1}, k \ll n$,

$$P \geq \int\lambda_n\left\{i \in \mathcal{N}_n \;:\; H\left(\omega|_{i+k}\right) > \frac{1}{2}\dim\mu - \epsilon\right\}\,d\beta(\omega) - \epsilon. \tag{4.10}$$

Be Lemma 4.13, for all $(i,\omega) \in \mathcal{N}_n \times \Lambda^{\mathbb{N}}$ we have

$$H\left(\omega|_{i+k}\right) \geq \frac{1}{m}H\left(S_{(\sigma^i\omega)|_k}\pi_{\ell(i,\omega)}\varphi_{(\sigma^i\omega)|_k}\mu, \mathcal{D}_m\right) - \epsilon/2, \tag{4.11}$$

where

$$\ell(i,\omega) := \frac{z}{(\psi\circ\varphi_u)'(\Pi\omega)}\mathbb{R}\alpha_i(\omega)^{-1} \in \mathbb{RP}^1.$$

By Lemma 4.12, for each $v \in \Lambda^k$ there exists $w_v\mathbb{R} \in \mathbb{RP}^1$ such that

$$\frac{1}{m}H\left(S_v\pi_{w\mathbb{R}}\varphi_v\mu, \mathcal{D}_m\right) > \frac{1}{2}\dim\mu - \epsilon/2 \text{ for all } w\mathbb{R} \in \mathbb{RP}^1 \setminus B\left(w_v\mathbb{R}, \epsilon\right). \tag{4.12}$$

Let $h : \Lambda^{\mathbb{N}} \to \mathbb{RP}^1$ be defined by $h(\omega) = w_{\omega|_k}\mathbb{R}$ for $\omega \in \Lambda^{\mathbb{N}}$. Note that since $\epsilon^{-1}, m, k \ll n$, we may assume that n is large with respect to h. From (4.11) and (4.12),

$$H\left(\omega|_{i+k}\right) > \frac{1}{2}\dim\mu - \epsilon \text{ for all } (i,\omega) \in \mathcal{N}_n \times \Lambda^{\mathbb{N}} \text{ such that } \ell(i,\omega) \notin B\left(h(\sigma^i\omega), \epsilon\right).$$

Hence, by (4.10) and the definition of the lines $\ell(i,\omega)$,

$$\begin{aligned} P &\geq \int \lambda_n\left\{i \in \mathcal{N}_n \,:\, \ell(i,\omega) \notin B\left(h(\sigma^i\omega),\epsilon\right)\right\}\,d\beta(\omega) - \epsilon \\ &= \int \lambda_n\left\{i \in \mathcal{N}_n \,:\, d\left(\frac{z}{(\psi\circ\varphi_u)'(\Pi\omega)}\mathbb{R}, \alpha_i(\omega)h(\sigma^i\omega)\right) > \epsilon\right\}\,d\beta(\omega) - \epsilon. \end{aligned}$$

From this, by Corollary 4.10, since n is large with respect to h, and since $\epsilon < \delta/3$, it follows that $P \geq \delta/2$, which completes the proof of the proposition. □

4.4. **Proof of Propositions 4.1 and 4.2.** First, we prove Proposition 4.2, which we restate here for convenience.

Proposition. *Suppose that* $\dim\mu < 2$. *Then there exists* $0 < \gamma < 1$ *such that for every* $\epsilon > 0$, $n \geq N(\epsilon) \geq 1$, $z\mathbb{R} \in \mathbb{RP}^1$, $u \in \Lambda^*$, *and* $\psi \in \mathcal{F}_\epsilon$,

$$\frac{1}{n}H\left(S_u\pi_{z\mathbb{R}}\psi\varphi_u\mu, \mathcal{D}_n\right) \geq \dim\mu - 1 + \gamma.$$

Proof. Let $0 < \delta < 1$ be as obtained in Proposition 4.11, and let $0 < \epsilon < 1$ and $l, m, n \in \mathbb{Z}_{>0}$ be with

$$\delta^{-1} \ll l \ll \epsilon^{-1} \ll m \ll n.$$

Let $M = M(l) \in \mathbb{Z}_{>0}$ be as obtained in Lemma 2.8. Since $l \ll \epsilon^{-1}$, we may assume that $M \ll \epsilon^{-1}$. Fix $z\mathbb{R} \in \mathbb{RP}^1$, $u \in \Lambda^*$ and $\psi \in \mathcal{F}_\epsilon$, and set

$$H := \frac{1}{n}H\left(S_u\pi_{z\mathbb{R}}\psi\varphi_u\mu, \mathcal{D}_n\right).$$

Set $n' := \lfloor n/M \rfloor$, and let $\mathcal{V}$ denote the set of $v \in \Lambda^*$ such that

$$\frac{1}{m}H\left(S_{uv}\pi_{z\mathbb{R}}\psi\varphi_{uv}\mu, \mathcal{D}_m\right) > \frac{1}{2}\dim\mu - \epsilon.$$

By Proposition 4.11 and since $\delta^{-1}, \epsilon^{-1}, l, M, m \ll n$,

$$\mathbb{P}_{l\leq i\leq n'l+l-1}\left\{\mathbf{U}_i \in \mathcal{V}\right\} > \delta/2.$$

Hence, there exists $0 \leq j < l$ such that

$$\mathbb{P}_{1\leq i\leq n'}\left\{\mathbf{U}_{j+li} \in \mathcal{V}\right\} > \delta/2. \tag{4.13}$$

Setting

$$\rho := \mathbb{P}_{1\leq i\leq n}\left\{\mathbf{I}\left(j,l;i\right) \in \mathcal{V}\right\},$$

it follows from (4.13), by Lemma 2.8, and since $M \ll n$, that $\rho > \delta/(4M)$.

By Lemma 2.3,

$$\operatorname{diam}\left(\operatorname{supp}\left(S_v \pi_{z\mathbb{R}} \psi \varphi_v \mu\right)\right) = O\left(1/\epsilon\right) \text{ for all } v \in \Lambda^*. \tag{4.14}$$

From this, by [9, Lemma 3.4], and since $\epsilon^{-1}, m \ll n$,

$$H \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m} H\left(S_u \pi_{z\mathbb{R}} \psi \varphi_u \mu, \mathcal{D}_{i+m} \mid \mathcal{D}_i\right)\right) - \epsilon.$$

By (2.8), we have $\varphi_u \mu = \mathbb{E}\left(\varphi_{u\mathbf{I}(j,l;i)} \mu\right)$ for each $i \geq 1$. Hence, from the last formula, by the concavity of conditional entropy, from (2.7), and since $l, \epsilon^{-1} \ll m$,

$$H \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m} H\left(S_u S_{\mathbf{I}(j,l;i)} \pi_{z\mathbb{R}} \psi \varphi_{u\mathbf{I}(j,l;i)} \mu, \mathcal{D}_m \mid \mathcal{D}_0\right)\right) - 2\epsilon.$$

From this, by bounded distortion, from (4.14), and since $\epsilon^{-1} \ll m$,

$$H \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m} H\left(S_{u\mathbf{I}(j,l;i)} \pi_{z\mathbb{R}} \psi \varphi_{u\mathbf{I}(j,l;i)} \mu, \mathcal{D}_m\right)\right) - 3\epsilon.$$

Now, from the last formula, by Lemma 4.4, and since $\rho > \delta/(4M)$,

$$\begin{aligned} H &\geq \rho\left(\frac{1}{2}\dim\mu - \epsilon\right) + (1-\rho)\left(\dim\mu - 1 - \epsilon\right) - 3\epsilon \\ &\geq \dim\mu - 1 + \frac{\delta}{4M}\left(1 - \frac{1}{2}\dim\mu\right) - 4\epsilon. \end{aligned}$$

Since $\dim\mu < 2$ and $\delta^{-1}, M \ll \epsilon^{-1}$, this completes the proof of the proposition. □

We can now prove Proposition 4.1, which is the following statement.

Proposition. *Suppose that* $\dim\mu < 2$. *Then there exists* $\gamma > 0$ *such that for every* $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$, $n \geq 1$, *and* $\psi \in \mathcal{F}_\epsilon$, *if we set* $\theta := \psi\mu$, *then*

$$\mathbb{P}\left\{\inf_{w\mathbb{R}\in\mathbb{RP}^1} \frac{1}{m} H\left(\pi_{w\mathbb{R}} \theta_{z,n}, \mathcal{D}_{n+m}\right) > \dim\mu - 1 + \gamma\right\} > 1 - \epsilon.$$

Proof. Let $0 < \gamma < 1$ be as obtained in Proposition 4.2, let $0 < \epsilon, \delta < 1$ and $k, m, n \in \mathbb{Z}_{>0}$ be with

$$\gamma^{-1} \ll \epsilon^{-1} \ll \delta^{-1} \ll k \ll m,$$

and fix $\psi \in \mathcal{F}_\epsilon$.

Set $B := \cup_{D\in\mathcal{D}_n^{\mathbb{C}}} (\partial D)^{(2^{-n}\delta)}$. By Proposition 3.4 and since $\epsilon^{-1} \ll \delta^{-1}$, we may assume that $\psi\mu(B) < \epsilon^2$. Let $\mathcal{U}$ be the set of words $u \in \Psi_{n+k}$ such that $\psi \circ \varphi_u(\Omega) \subset D$ for some $D \in \mathcal{D}_n^{\mathbb{C}}$. By Lemma 2.3 and (2.7),

$$\operatorname{diam}\left(\psi \circ \varphi_u(\Omega)\right) = O\left(\epsilon^{-1} 2^{-n-k}\right) \text{ for } u \in \Psi_{n+k}.$$

From this and since $\epsilon^{-1}, \delta^{-1} \ll k$, it follows that $\psi \circ \varphi_u(\Omega) \subset B$ for $u \in \Psi_{n+k} \setminus \mathcal{U}$. Thus, by (2.8),

$$\sum_{u\in\Psi_{n+k}\setminus\mathcal{U}} p_u \leq \sum_{u\in\Psi_{n+k}} p_u \cdot \psi\varphi_u\mu(B) = \psi\mu(B) < \epsilon^2,$$

which implies,

$$\epsilon^2 > \sum_{D\in\mathcal{D}_n^{\mathbb{C}}} \psi\mu(D) \sum_{u\in\Psi_{n+k}\setminus\mathcal{U}} p_u \cdot \frac{\psi\varphi_u\mu(D)}{\psi\mu(D)}. \tag{4.15}$$

Let $\mathcal{E}$ be the set of $D \in \mathcal{D}_n^{\mathbb{C}}$ such that $\psi\mu(D) > 0$ and

$$\frac{1}{\psi\mu(D)} \sum_{u \in \Psi_{n+k} \setminus \mathcal{U}} p_u \cdot \psi\varphi_u\mu(D) < \epsilon.$$

By (4.15), we have $\psi\mu(\cup\mathcal{E}) > 1 - \epsilon$.

Fix $D \in \mathcal{E}$ and let $\mathcal{U}_D$ be the set of $u \in \Psi_{n+k}$ such that $\psi \circ \varphi_u(\Omega) \subset D$. By (2.8),

$$(\psi\mu)_D = \frac{1}{\psi\mu(D)} \sum_{u \in \Psi_{n+k}} p_u \cdot \psi\varphi_u\mu(D) \cdot (\psi\varphi_u\mu)_D . \tag{4.16}$$

Additionally, from $D \in \mathcal{E}$ and since $\psi\varphi_u\mu(D) = 0$ for $u \in \mathcal{U} \setminus \mathcal{U}_D$,

$$\frac{1}{\psi\mu(D)} \sum_{u \in \mathcal{U}_D} p_u \cdot \psi\varphi_u\mu(D) > 1 - \epsilon. \tag{4.17}$$

Fix $w\mathbb{R} \in \mathbb{RP}^1$. Given $u \in \mathcal{U}_D$, we have $(\psi\varphi_u\mu)_D = \psi\varphi_u\mu$. Hence, from $u \in \Psi_{n+k}$, (2.7), and $\gamma^{-1}, k \ll m$,

$$\frac{1}{m} H\left(\pi_{w\mathbb{R}} (\psi\varphi_u\mu)_D , \mathcal{D}_{n+m}\right) \geq \frac{1}{m} H\left(S_u \pi_{w\mathbb{R}} \psi\varphi_u\mu, \mathcal{D}_m\right) - \gamma/2.$$

From this and by Proposition 4.2,

$$\frac{1}{m} H\left(\pi_{w\mathbb{R}} (\psi\varphi_u\mu)_D , \mathcal{D}_{n+m}\right) \geq \dim\mu - 1 + \gamma/2 \text{ for all } u \in \mathcal{U}_D.$$

Thus, by concavity of entropy, (4.16), (4.17), and $\gamma^{-1} \ll \epsilon^{-1}$,

$$\begin{aligned} \frac{1}{m} H\left(\pi_{w\mathbb{R}} (\psi\mu)_D , \mathcal{D}_{n+m}\right) &\geq (1-\epsilon)\left(\dim\mu - 1 + \gamma/2\right) \\ &> \dim\mu - 1 + \gamma/4, \end{aligned}$$

which holds for all $D \in \mathcal{E}$ and $w\mathbb{R} \in \mathbb{RP}^1$. Since $\psi\mu(\cup\mathcal{E}) > 1 - \epsilon$, this completes the proof of the proposition. □

5. An entropy increase result

The purpose of this section is to prove the following theorem. The proof resembles that of [20, Theorem 3.1], except that here we must take into account the phenomenon of saturation, which does not occur on the real line and was addressed in the previous section.

Theorem 5.1. *Suppose that* $\dim\mu < 2$. *Then for each* $k \in \mathbb{Z}_{>0}$ *and* $0 < \epsilon < 1$ *there exists* $\rho = \rho(k, \epsilon) > 0$ *such that the following holds for all* $n \geq N(k, \epsilon) \geq 1$. *Let* $\nu \in \mathcal{M}(\mathcal{P}_k)$ *be such that* $\|q\|_2 \leq \epsilon^{-1}$ *and* $|q'(z)| \geq \epsilon$ *for all* $q \in \operatorname{supp}(\nu)$ *and* $z \in \mathbb{D}$. *Suppose also that* $\frac{1}{n} H(\nu, \mathcal{D}_n) \geq \epsilon$, *and let* $\psi \in \mathcal{F}_\epsilon$ *be with* $\psi(\Omega) \subset \mathbb{D}$. *Then* $\frac{1}{n} H\left(\nu.(\psi\mu), \mathcal{D}_n\right) \geq \dim\mu + \rho$.

5.1. Entropy growth under convolution in $\mathbb{C}$.

The proof of Theorem 5.1 relies on the following theorem, which follows directly from [11, Theorem 2.8]; a complete derivation is given in [21, Section 5.1]. For $\theta, \xi \in \mathcal{M}(\mathbb{C})$, we write $\theta * \xi$ for their convolution, that is, the push-forward of $\theta \times \xi$ under the map $(x, y) \mapsto x + y$.

Theorem 5.2. *For every $0 < \epsilon < 1$, $m \geq 1$ and $0 < \eta < \eta(\epsilon)$, there exists $\delta = \delta(\epsilon, m, \eta) > 0$, such that for all $n \geq N(\epsilon, m, \eta) \geq 1$ the following holds. Let $i \in \mathbb{Z}_{>0}$ and $\xi, \theta \in \mathcal{M}(\mathbb{C})$ be such that*

$$\operatorname{diam}(\operatorname{supp}(\xi)), \operatorname{diam}(\operatorname{supp}(\theta)) \leq \epsilon^{-1} 2^{-i},$$

$$\mathbb{P}_{i \leq j \leq i+n} \left\{ \frac{1}{m} H\left(\theta_{z,j}, \mathcal{D}_{j+m}\right) < 2 - \epsilon \right\} > 1 - \eta,$$

$$\mathbb{P}_{i \leq j \leq i+n} \left\{ \inf_{w\mathbb{R} \in \mathbb{RP}^1} \frac{1}{m} H\left(\pi_{w\mathbb{R}} \theta_{z,j}, \mathcal{D}_{j+m}\right) > \frac{1}{m} H\left(\theta_{z,j}, \mathcal{D}_{j+m}\right) - 1 + \epsilon \right\} > 1 - \eta,$$

and

$$\frac{1}{n} H\left(\xi, \mathcal{D}_{i+n}\right) > \epsilon.$$

Then,

$$\frac{1}{n} H\left(\xi * \theta, \mathcal{D}_{i+n}\right) \geq \frac{1}{n} H\left(\theta, \mathcal{D}_{i+n}\right) + \delta.$$

5.2. **Uniform entropy dimension.** By Lemma 4.3 and (2.7), for every $\epsilon > 0$, $m \geq M(\epsilon) \geq 1$, $n \geq 1$, $u \in \Psi_n$, and $\psi \in \mathcal{F}_\epsilon$, we have

$$\frac{1}{m} H\left(\psi \varphi_u \mu, \mathcal{D}_{n+m}\right) \geq \dim \mu - \epsilon.$$

The following proposition, whose proof we omit, follows from this and from Proposition 3.4 by an argument similar to that in the proof of [20, Proposition 3.6].

Proposition 5.3. *For each $0 < \epsilon < 1$, $m \geq M(\epsilon) \geq 1$, $n \geq 1$, and $\psi \in \mathcal{F}_\epsilon$, if we set $\theta := \psi\mu$, then*

$$\mathbb{P}\left\{ \frac{1}{m} H\left(\theta_{z,n}, \mathcal{D}_{n+m}\right) > \dim \mu - \epsilon \right\} > 1 - \epsilon.$$

Roughly speaking, and in the terminology of [9], the following proposition states that holomorphic images of μ have uniform entropy dimension $\dim \mu$. It follows easily from Proposition 5.3 by an argument identical to the one given in the proof of [20, Proposition 3.3]. We therefore omit the proof.

Proposition 5.4. *For each $0 < \epsilon < 1$, $m \geq M(\epsilon) \geq 1$, $n \geq N(\epsilon, m)$, and $\psi \in \mathcal{F}_\epsilon$, if we set $\theta := \psi\mu$, then*

$$\mathbb{P}_{1 \leq i \leq n} \left\{ \left| \frac{1}{m} H\left(\theta_{z,i}, \mathcal{D}_{i+m}\right) - \dim \mu \right| < \epsilon \right\} > 1 - \epsilon.$$

5.3. **Additional preparations for the proof of Theorem 5.1.** The proof of the following lemma is similar to that of [12, Lemma 6.9] and is therefore omitted.

Lemma 5.5. *Let $k \in \mathbb{Z}_{>0}$, $R > 1$, $\nu \in \mathcal{M}(\mathcal{P}_k)$ and $\theta \in \mathcal{M}(\mathbb{D})$ be given. Suppose that $\|q\|_2 \leq R$ for $q \in \operatorname{supp}(\nu)$. Then for every $n \geq m \geq 1$,*

$$\frac{1}{n} H\left(\nu.\theta, \mathcal{D}_n\right) \geq \mathbb{E}_{1 \leq i \leq n} \left(\frac{1}{m} H\left(\nu_{q,i}.\theta_{z,i}, \mathcal{D}_{i+m}\right) \right) - O_{k,R}\left(\frac{1}{m} + \frac{m}{n} \right).$$

Writing F for the map taking $(q, z) \in \mathcal{P}_k \times \mathbb{D}$ to $q(z)$, the (complex) differential of F at a point (q, z) is given by $dF_{(q,z)}(r, w) = r(z) + q'(z)w$. Using this, the proof of the following linearization lemma is almost identical to that of [2, Lemma 4.2], so we omit it.

Lemma 5.6. *For every $\epsilon > 0$, $k \in \mathbb{Z}_{>0}$, $R > 1$, $m \geq M(\epsilon) \geq 1$, and $0 < \delta < \delta(\epsilon, k, R, m)$, the following holds. Let $q \in \mathcal{P}_k$, $z \in \mathbb{D}$, $\nu \in \mathcal{M}(\mathcal{P}_k)$ and $\theta \in \mathcal{M}(\mathbb{D})$ be such that $\|q\|_2 \leq R$, $\|r - q\|_2 \leq \delta$ for $r \in \mathrm{supp}(\nu)$, and $|w - z| \leq \delta$ for $w \in \mathrm{supp}(\theta)$. Then,*

$$\left| \frac{1}{m} H\left(\nu.\theta, \mathcal{D}_{m-\log\delta}\right) - \frac{1}{m} H\left((\nu.z) * \left(S_{q'(z)}\theta\right), \mathcal{D}_{m-\log\delta}\right) \right| < \epsilon.$$

Finally, the following lemma is obtained by an argument similar to that used in the proof of [20, Lemma 3.10], and we again omit the details.

Lemma 5.7. *For every $0 < \epsilon < 1$ and $k \in \mathbb{Z}_{>0}$ there exists $\epsilon_0 = \epsilon_0(\epsilon, k) > 0$ such that for all $m \geq M(\epsilon, k) \geq 1$ and $n \geq N(\epsilon, k, m)$ the following holds. Let $\nu \in \mathcal{M}(\mathcal{P}_k)$ be such that $\|q\|_2 \leq \epsilon^{-1}$ for $q \in \mathrm{supp}(\nu)$ and $\frac{1}{n}H(\nu, \mathcal{D}_n) \geq \epsilon$. Let $\psi \in \mathcal{F}_\epsilon$ be with $\psi(\Omega) \subset \mathbb{D}$, and set $\theta := \psi\mu$. Then,*

$$\int \mathbb{P}_{1\leq i\leq n}\left\{\frac{1}{m}H\left(\left(\nu_{q,i}\right).z, \mathcal{D}_{i+m}\right) > \epsilon_0\right\} d\theta(z) > \epsilon_0. \tag{5.1}$$

5.4. **Proof of the entropy increase result.**

Proof of Theorem 5.1. Suppose that $\dim\mu < 2$, let $0 < \gamma < 1$ be as obtained in Proposition 4.1, and let $k, l, m, n \in \mathbb{Z}_{>0}$ and $0 < \epsilon, \epsilon_0, \eta, \rho, \delta < 1$ be with,

$$(2 - \dim\mu)^{-1}, \gamma, k, \epsilon^{-1} \ll \epsilon_0^{-1} \ll \eta^{-1} \ll l \ll \rho^{-1} \ll \delta^{-1} \ll m \ll n. \tag{5.2}$$

Let $\nu \in \mathcal{M}(\mathcal{P}_k)$ be such that $\|q\|_2 \leq \epsilon^{-1}$ and $|q'(z)| \geq \epsilon$ for $q \in \mathrm{supp}(\nu)$ and $z \in \mathbb{D}$, and such that $\frac{1}{n}H(\nu, \mathcal{D}_n) \geq \epsilon$. Let $\psi \in \mathcal{F}_\epsilon$ be with $\psi(\Omega) \subset \mathbb{D}$, and set $\theta := \psi\mu$.

By Lemma 5.5,

$$\frac{1}{n}H(\nu.\theta, \mathcal{D}_n) + \delta \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m}H\left(\nu_{q,i}.\theta_{z,i}, \mathcal{D}_{i+m}\right)\right).$$

Hence, by Lemma 5.6,

$$\frac{1}{n}H(\nu.\theta, \mathcal{D}_n) + 2\delta \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m}H\left((\nu_{q,i}.z) * \left(S_{q'(z)}\theta_{z,i}\right), \mathcal{D}_{i+m}\right)\right).$$

Thus, since $|q'(z)| \geq \epsilon$ for $q \in \mathrm{supp}(\nu)$ and $z \in \mathbb{D}$,

$$\frac{1}{n}H(\nu.\theta, \mathcal{D}_n) + 3\delta \geq \mathbb{E}_{1\leq i\leq n}\left(\frac{1}{m}H\left(\left(S^{-1}_{q'(z)}\nu_{q,i}\right).z * \theta_{z,i}, \mathcal{D}_{i+m}\right)\right). \tag{5.3}$$

Write $\Gamma := \lambda_n \times \theta \times \nu$, where λ_n is defined in Section 2.1. Let E_1 be the set of all $(i, z, q) \in \mathcal{N}_n \times \mathbb{D} \times \mathrm{supp}(\nu)$ such that $\frac{1}{m}H(\theta_{z,i}, \mathcal{D}_{i+m}) \geq \dim\mu - \delta$. By Proposition 5.3, we may assume that $\Gamma(E_1) > 1 - \delta$. Also, by [9, Corollary 4.10],

$$\frac{1}{m}H\left(\left(S^{-1}_{q'(z)}\nu_{q,i}\right).z * \theta_{z,i}, \mathcal{D}_{i+m}\right) \geq \dim\mu - 2\delta \text{ for } (i, z, q) \in E_1. \tag{5.4}$$

Let E_2 be the set of all $(i, z, q) \in E_1$ such that

$$\mathbb{P}_{i\leq j\leq i+m}\left\{\frac{1}{l}H\left(\left(\theta_{z,i}\right)_{w,j}, \mathcal{D}_{j+l}\right) < 1 + \frac{1}{2}\dim\mu\right\} > 1 - \eta, \tag{5.5}$$

$$\mathbb{P}_{i\leq j\leq i+m}\left\{\begin{array}{c}\inf\limits_{u\mathbb{R}\in\mathbb{RP}^1} \frac{1}{l}H\left(\pi_{u\mathbb{R}}\left(\theta_{z,i}\right)_{w,j}, \mathcal{D}_{j+l}\right) \\ > \frac{1}{l}H\left(\left(\theta_{z,i}\right)_{w,j}, \mathcal{D}_{j+l}\right) - 1 + \gamma/2\end{array}\right\} > 1 - \eta, \tag{5.6}$$

and

$$\frac{1}{m}H\left(\left(\nu_{q,i}\right).z,\mathcal{D}_{i+m}\right)>\epsilon_0. \tag{5.7}$$

By Propositions 4.1 and 5.4, Lemma 5.7, [11, Lemma 2.7], and since $\dim\mu<2$, we may assume that $\Gamma(E_2)>\epsilon_0$.

Let $(i,z,q)\in E_2$ and set $\xi:=\left(S_{q'(z)}^{-1}\nu_{q,i}\right).z$. We next want to apply Theorem 5.2 in order to obtain entropy increase for the convolution $\xi*\theta_{z,i}$. Let $q_1,q_2\in\operatorname{supp}(\nu_{q,i})$ be given. Since $z\in\mathbb{D}$, $|q'(z)|\geq\epsilon$, and q_1 and q_2 belong to the same atom of $\mathcal{D}_i^{\mathcal{P}_k}$,

$$|q'(z)|^{-1}\cdot|q_1(z)-q_2(z)|\leq\epsilon^{-1}(k+1)2^{1-i}.$$

From this and since $\theta_{z,i}$ is supported on a single atom of $\mathcal{D}_i^{\mathbb{C}}$,

$$\operatorname{diam}\left(\operatorname{supp}\left(\xi\right)\right),\operatorname{diam}\left(\operatorname{supp}\left(\theta_{z,i}\right)\right)=O_{k,\epsilon}\left(2^{-i}\right). \tag{5.8}$$

We have $\|q\|_2\leq\epsilon^{-1}$, which implies $|q'(z)|=O_{k,\epsilon}(1)$. Thus, from (5.7) and since $k,\epsilon^{-1},\epsilon_0^{-1}\ll m$,

$$\frac{1}{m}H\left(\xi,\mathcal{D}_{i+m}\right)>\epsilon_0/2.$$

From this, (5.2), (5.5), (5.6), (5.8), and Theorem 5.2,

$$\frac{1}{m}H\left(\xi*\theta_{z,i},\mathcal{D}_{i+m}\right)\geq\frac{1}{m}H\left(\theta_{z,i},\mathcal{D}_{i+m}\right)+\rho.$$

Hence, since $E_2\subset E_1$, we have thus proven that

$$\frac{1}{m}H\left(\left(S_{q'(z)}^{-1}\nu_{q,i}\right).z*\theta_{z,i},\mathcal{D}_{i+m}\right)\geq\dim\mu+\rho-\delta\text{ for }(i,z,q)\in E_2. \tag{5.9}$$

Now, from (5.3), (5.4) and (5.9),

$$\frac{1}{n}H\left(\nu.\theta,\mathcal{D}_n\right)+3\delta\geq\Gamma\left(E_1\setminus E_2\right)\left(\dim\mu-2\delta\right)+\Gamma\left(E_2\right)\left(\dim\mu+\rho-\delta\right).$$

Thus, since $\Gamma(E_1)>1-\delta$ and $\Gamma(E_2)>\epsilon_0$,

$$\frac{1}{n}H\left(\nu.\theta,\mathcal{D}_n\right)\geq\dim\mu+\epsilon_0\rho-O(\delta).$$

By (5.2), this completes the proof of the theorem. □

6. Proof of the main result

In this section we prove our main result, Theorem 1.5. The proof is an adaptation of the proof of the main result of [20] and largely follows it. Nevertheless, for the reader's convenience, we shall provide full details of the main argument.

The proof of the following preliminary lemma is almost identical to that of [20, Lemma 4.1] and is therefore omitted. Recall from Section 2.8 that $\{\beta_\omega\}_{\omega\in\Lambda^{\mathbb{N}}}\subset\mathcal{M}\left(\Lambda^{\mathbb{N}}\right)$ denotes the disintegration of β with respect to $\Pi^{-1}\mathcal{B}_{\mathbb{C}}$. Also, recall that for each $n\geq1$, we defined $\Pi_n:\Lambda^{\mathbb{N}}\to\mathcal{O}(\Omega)$ by $\Pi_n(\omega)=\varphi_{\omega|_n}$ for $\omega\in\Lambda^{\mathbb{N}}$.

Lemma 6.1. *For β-a.e. ω,*

$$\lim_{n\to\infty}\frac{1}{n}H\left(\left(\Pi_n\beta_\omega\right).\mu,\mathcal{D}_{\chi n}\right)=0.$$

We shall also need the following lemma.

Lemma 6.2. *There exists $C > 1$ such that for all $u \in \Lambda^*$ the following holds. Let $\psi : \mathbb{C} \to \mathbb{C}$ be the unique homothetic map such that $\psi'(0) > 0$, $\psi(\varphi_u(0)) = 0$, and*

$$\sup \{|\psi(\varphi_u(z))| \,:\, z \in \Omega\} = 1.$$

Let $r_\psi > 0$ and $b_\psi \in \mathbb{C}$ be with $\psi(z) = r_\psi z + b_\psi$ for $z \in \mathbb{C}$. Then, $C^{-1} \le r_\psi |\varphi_u'(0)| \le C$.

Proof. Let $C > 1$ be a large global constant depending only on Φ and Ω. In particular, since the bounded domain Ω contains 0, we may assume that $C^{-1} \le |z| \le C$ for all $z \in \partial\Omega$.

Let $u \in \Lambda^*$, and let $\psi : \mathbb{C} \to \mathbb{C}$ be as in the statement of the lemma. Since $\overline{\Omega}$ is compact and by the maximum modulus principle, there exists $w \in \partial\Omega$ such that $|\psi(\varphi_u(w))| = 1$. We have,

$$1 = |\psi(\varphi_u(w)) - \psi(\varphi_u(0))| = r_\psi |\varphi_u(w) - \varphi_u(0)|.$$

Moreover, by Lemma 2.3,

$$C^{-1} |\varphi_u'(0)| |w| \le |\varphi_u(w) - \varphi_u(0)| \le C |\varphi_u'(0)| |w|.$$

Combining all of this, we obtain $C^{-2} \le r_\psi |\varphi_u'(0)| \le C^2$, which completes the proof of the lemma. ∎

We can now begin the proof of our main result.

Proof of Theorem 1.5. Suppose that the assumptions made in the theorem are all satisfied, and assume by contradiction that $\dim \mu < \min\{2, H(p)/\chi\}$. Recall that for $l \ge 1$, we denote by $\mathcal{C}_l$ the partition of $\Lambda^{\mathbb{N}}$ into level-l cylinder sets. Set $\Delta' := H(\beta, \mathcal{C}_1 \mid \Pi^{-1}\mathcal{B}_{\mathbb{C}})$, where the right-hand side denotes the conditional entropy of $\mathcal{C}_1$ given $\Pi^{-1}\mathcal{B}_{\mathbb{C}}$ with respect to β. By [8, Theorem 2.8], we have $\dim \mu = (H(p) - \Delta')/\chi$. Thus, from $\dim \mu < H(p)/\chi$, it follows that $\Delta' > 0$. Moreover, by [8, Proposition 4.10],

$$\lim_{l \to \infty} \frac{1}{l} H(\beta_\omega, \mathcal{C}_l) = \Delta' \text{ for } \beta\text{-a.e. } \omega. \tag{6.1}$$

Since Φ is exponentially separated, there exists $0 < c < 1$ as in Definition 1.1. Set $\Delta := \min\{\Delta', 1\}/2$, and let $C > 1$ be a large global constant depending only on Φ, Ω and Ω_0. Let $0 < \rho, \delta < 1$ and $M, k, n \in \mathbb{Z}_{>0}$ be such that

$$\Delta^{-1}, c^{-1}, C \ll M \ll k \ll \rho^{-1} \ll \delta^{-1} \ll n. \tag{6.2}$$

Set $n' = \lfloor n\Delta / (2 \log |\Lambda|) \rfloor$. By exponential separation and the choice of c, we may clearly assume that

$$\|\varphi_{u_1} - \varphi_{u_2}\|_\Omega \ge c^{n+n'} \text{ for all distinct } u_1, u_2 \in \Lambda^{n+n'}. \tag{6.3}$$

By (2.5) and $\delta^{-1} \ll n$, we may also assume that

$$\dim \mu + \delta \ge \frac{1}{Mn} H\left(\mu, \mathcal{D}_{Mn+\chi(n+n')} \mid \mathcal{D}_{\chi(n+n')}\right). \tag{6.4}$$

By (2.1) and since $\beta = \int \beta_\omega \, d\beta(\omega)$,

$$\mu = (\Pi_{n+n'}\beta).\mu = \int (\Pi_{n+n'}\beta_\omega).\mu \, d\beta(\omega).$$

Hence, from (6.4) and by the concavity of conditional entropy,

$$\dim \mu + \delta \ge \int \frac{1}{Mn} H\left((\Pi_{n+n'}\beta_\omega).\mu, \mathcal{D}_{Mn+\chi(n+n')} \mid \mathcal{D}_{\chi(n+n')}\right) \, d\beta(\omega).$$

Together with Lemma 6.1, this gives

$$\dim \mu + 2\delta \geq \int \frac{1}{Mn} H\left(\left(\Pi_{n+n'}\beta_\omega\right).\mu, \mathcal{D}_{Mn+\chi(n+n')}\right)\, d\beta(\omega).$$

Thus, setting

$$g(\omega) := \frac{1}{Mn} H\left(\left(\Pi_{n+n'}\beta_\omega\right).\mu, \mathcal{D}_{Mn+\chi(n+n')}\right) \text{ for } \omega \in \Lambda^{\mathbb{N}},$$

we have

$$\dim \mu + 2\delta \geq \int g(\omega)\, d\beta(\omega). \tag{6.5}$$

Recall that for $\mathcal{U} \subset \Lambda^*$, we write $[\mathcal{U}]$ in place of $\cup_{u\in\mathcal{U}}[u]$. For $l \in \mathbb{Z}_{>0}$, let $\mathcal{U}_l$ be the set of words $u \in \Lambda^l$ such that $2^{-l(\chi+\delta)} \leq |\varphi_u'(0)| \leq 2^{-l(\chi-\delta)}$. Let E be the set of $\omega \in \Lambda^{\mathbb{N}}$ such that $\mathrm{supp}(\beta_\omega) \subset \Pi^{-1}(\Pi\omega)$, $\frac{1}{n}H(\beta_\omega, \mathcal{C}_n) > \Delta$, $\beta_\omega([\mathcal{U}_n]) > 1-\delta$, and $\sigma^n\beta_\omega([\mathcal{U}_{n'}]) > 1-\delta$. By (6.1) and (2.6), by basic properties of disintegrations, and since $\Delta^{-1}, \delta^{-1} \ll n$, we may assume that $\beta(E) > 1-\delta$. In what follows, fix $\omega \in E$.

By the definition of $\mathcal{U}_n$, there exists a partition $\{\mathcal{U}_{n,j}\}_{j\in J}$ of $\mathcal{U}_n$ such that $|J| \leq 3\delta n$ and,

$$\left|\varphi_{u_1}'(0)\right| \leq 2\left|\varphi_{u_2}'(0)\right| \text{ for all } j \in J \text{ and } u_1, u_2 \in \mathcal{U}_{n,j}. \tag{6.6}$$

For $j \in J$ and $u \in \mathcal{U}_{n'}$, set $F_{j,u} := [\mathcal{U}_{n,j}] \cap \sigma^{-n}[u]$ and

$$g(\omega, j, u) := \frac{1}{Mn} H\left(\left(\Pi_{n+n'}\left(\beta_\omega\right)_{F_{j,u}}\right).\mu, \mathcal{D}_{Mn+\chi(n+n')}\right).$$

Since $\omega \in E$, there exist $0 \leq \delta' < 2\delta$ and $\theta \in \mathcal{M}\left(\Lambda^{\mathbb{N}}\right)$ such that

$$\beta_\omega = \sum_{(j,u)\in J\times\mathcal{U}_{n'}} \beta_\omega\left(F_{j,u}\right)\left(\beta_\omega\right)_{F_{j,u}} + \delta'\theta.$$

Thus, by concavity,

$$g(\omega) \geq \sum_{(j,u)\in J\times\mathcal{U}_{n'}} \beta_\omega\left(F_{j,u}\right) g(\omega, j, u). \tag{6.7}$$

Moreover, from $\frac{1}{n}H(\beta_\omega, \mathcal{C}_n) > \Delta$, by the convexity bound (see [9, Lemma 3.1]), and since the cardinality of $J \times \mathcal{U}_{n'}$ is at most $3\delta n|\Lambda|^{n'}$,

$$\Delta < \sum_{(j,u)\in J\times\mathcal{U}_{n'}} \beta_\omega\left(F_{j,u}\right) \frac{1}{n} H\left(\left(\beta_\omega\right)_{F_{j,u}}, \mathcal{C}_n\right) + \delta'\frac{1}{n}H(\theta, \mathcal{C}_n) + \frac{\log(3\delta n)}{n} + \frac{n'}{n}\log|\Lambda|.$$

Note that,

$$\frac{1}{n}H(\theta', \mathcal{C}_n) \leq \log|\Lambda| \text{ for all } \theta' \in \mathcal{M}\left(\Lambda^{\mathbb{N}}\right). \tag{6.8}$$

Hence, from the previous formula and the definition of n',

$$\Delta/3 < \sum_{(j,u)\in J\times\mathcal{U}_{n'}} \beta_\omega\left(F_{j,u}\right) \frac{1}{n} H\left(\left(\beta_\omega\right)_{F_{j,u}}, \mathcal{C}_n\right). \tag{6.9}$$

Let $\mathcal{Q}$ be the set of all $(j,u) \in J \times \mathcal{U}_{n'}$ such that $\frac{1}{n}H\left(\left(\beta_\omega\right)_{F_{j,u}}, \mathcal{C}_n\right) \geq \Delta/6$. From (6.8) and (6.9),

$$\frac{\Delta}{6\log|\Lambda|} < \sum_{(j,u)\in\mathcal{Q}} \beta_\omega\left(F_{j,u}\right). \tag{6.10}$$

Recall that $C > 1$ is a large global constant. For $v \in \mathcal{U}_n$ and $w \in \mathcal{U}_{n'}$, it follows by bounded distortion, the chain rule, and the definition of the sets $\mathcal{U}_l$, that

$$C^{-1}2^{-\delta(n+n')} \leq 2^{\chi(n+n')}\left|\varphi'_{vw}(0)\right| \leq C2^{\delta(n+n')}.$$

Thus, by Lemma 2.3, $S_{2^{\chi(n+n')}} \circ \varphi_{vw}$ is bi-Lipschitz with bi-Lipschitz constant at most $C^2 2^{\delta(n+n')}$. Moreover, given $(j,u) \in J \times \mathcal{U}_{n'}$ and setting $\xi := \left(\Pi_{n+n'}\left(\beta_\omega\right)_{F_{j,u}}\right)$, we have

$$\operatorname{supp}(\xi) \subset \left\{\varphi_{vw} \,:\, v \in \mathcal{U}_n \text{ and } w \in \mathcal{U}_{n'}\right\}.$$

From these facts, by concavity, from Lemma 2.6, by (2.5), and since $C, \delta^{-1} \ll n$,

$$g(\omega,j,u) \geq \int \frac{1}{Mn} H\left(S_{2^{\chi(n+n')}}\varphi\mu, \mathcal{D}_{Mn}\right)\, d\xi(\varphi) - O(1/n) \geq \dim\mu - O(\delta).$$

We have thus shown that,

$$g(\omega,j,u) \geq \dim\mu - O(\delta) \text{ for } (j,u) \in J \times \mathcal{U}_{n'}. \tag{6.11}$$

Fix $(j,u) \in \mathcal{Q}$, and set $a := \varphi_u(0)$. Recall from Section 2.5 that the linear operator $P_{k,a} : \mathcal{O}(\Omega) \to \mathcal{P}_k$ is defined by sending $f \in \mathcal{O}(\Omega)$ to its k-th order Taylor polynomial at the point a. Let $v \in \mathcal{U}_{n,j}$ and $z \in \Omega$ be given. From Lemma 2.3 and since $u \in \mathcal{U}_{n'}$,

$$\left|\varphi_u(z) - a\right| \leq C\left|\varphi'_u(0)\right| \leq C2^{n'(\delta-\chi)}.$$

From this, since we can assume that $\delta < \chi/2$, by Lemma 2.5, and since $v \in \mathcal{U}_n$,

$$\begin{aligned}\left|\left(\varphi_v - P_{k,a}\varphi_v\right)\left(\varphi_u(z)\right)\right| &\leq C^{k+1}\left|\varphi'_v(0)\right|\left|\varphi_u(z) - a\right|^{k+1} \\ &\leq C^{2k+2}2^{(k+1)n'(\delta-\chi)}2^{n(\delta-\chi)}.\end{aligned}$$

Hence, from $\Delta^{-1}, C, M \ll k \ll n$, by the definition of n', and since $\delta < \chi/2$,

$$\left|\left(\varphi_v - P_{k,a}\varphi_v\right)\left(\varphi_u(z)\right)\right| \leq 2^{-Mn-\chi(n+n')} \text{ for } v \in \mathcal{U}_{n,j} \text{ and } z \in \Omega. \tag{6.12}$$

Note that

$$\left(\Pi_{n+n'}\left(\beta_\omega\right)_{F_{j,u}}\right).\mu = \left(\Pi_n\left(\beta_\omega\right)_{F_{j,u}}\right).\varphi_u\mu,$$

and also

$$\operatorname{supp}\left(\Pi_n\left(\beta_\omega\right)_{F_{j,u}}\right) = \left\{\varphi_v \,:\, v \in \mathcal{U}_{n,j} \text{ and } \beta_\omega\left([vu]\right) > 0\right\}. \tag{6.13}$$

Together with (6.12) and Lemma 2.7, this implies that

$$g(\omega,j,u) \geq \frac{1}{Mn}H\left(\left(P_{k,a}\Pi_n\left(\beta_\omega\right)_{F_{j,u}}\right).\varphi_u\mu, \mathcal{D}_{Mn+\chi(n+n')}\right) - \delta. \tag{6.14}$$

Let $\psi : \mathbb{C} \to \mathbb{C}$ be the unique homothetic map such that $\psi'(0) > 0$, $\psi(a) = 0$, and

$$\sup\left\{\left|\psi\left(\varphi_u(z)\right)\right| \,:\, z \in \Omega\right\} = 1.$$

Let $r_\psi > 0$ and $b_\psi \in \mathbb{C}$ be with $\psi(z) = r_\psi z + b_\psi$ for $z \in \mathbb{C}$. By Lemma 6.2 and since $u \in \mathcal{U}_{n'}$,

$$C^{-1}2^{n'(\chi-\delta)} \leq C^{-1}\left|\varphi'_u(0)\right|^{-1} \leq r_\psi \leq C\left|\varphi'_u(0)\right|^{-1} \leq C2^{n'(\chi+\delta)}. \tag{6.15}$$

Let $R_{\psi^{-1}} : \mathcal{P}_k \to \mathcal{P}_k$ be with $R_{\psi^{-1}}(q) := q \circ \psi^{-1}$ for $q \in \mathcal{P}_k$.

By (6.6) and $\mathcal{U}_{n,j} \subset \mathcal{U}_n$, there exists $2^{n(\chi-\delta)} \le \alpha \le 2^{n(\chi+\delta)}$ such that

$$\frac{1}{2} \le \alpha \left|\varphi_v'(0)\right| \le 2 \text{ for } v \in \mathcal{U}_{n,j}. \tag{6.16}$$

Set

$$\nu := S_{\alpha r_\psi} T_{-\Pi(\omega)} R_{\psi^{-1}} P_{k,a} \Pi_n \left(\beta_\omega\right)_{F_{j,u}},$$

where recall from Section 2.1 that for $q(X) \in \mathcal{P}_k$

$$T_{-\Pi(\omega)}\left(q(X)\right) = q(X) - \Pi(\omega) \;\text{ and }\; S_{\alpha r_\psi}\left(q(X)\right) = \alpha r_\psi \cdot q(X).$$

Note that,

$$\nu.\left(\psi\varphi_u\mu\right) = S_{\alpha r_\psi} T_{-\Pi(\omega)}\left(\left(P_{k,a}\Pi_n\left(\beta_\omega\right)_{F_{j,u}}\right).\varphi_u\mu\right).$$

Together with (6.14), $r_\psi \le C2^{n'(\chi+\delta)}$ and $\alpha \le 2^{n(\chi+\delta)}$, this gives

$$g(\omega, j, u) \ge \frac{1}{Mn} H\left(\nu.\left(\psi\varphi_u\mu\right), \mathcal{D}_{Mn}\right) - O(\delta). \tag{6.17}$$

Next, we aim to apply Theorem 5.1 to the entropy appearing on the right-hand side of the last inequality. We proceed to verify the conditions of the theorem. Let $v \in \mathcal{U}_{n,j}$ be with $\beta_\omega\left([vu]\right) > 0$, and set $q := S_{\alpha r_\psi} T_{-\Pi(\omega)} R_{\psi^{-1}} P_{k,a}\varphi_v$. Note that by (6.13), each polynomial in $\mathrm{supp}(\nu)$ is of this form. We have

$$\psi^{-1}(X) - a = r_\psi^{-1}X - b_\psi r_\psi^{-1} - a = r_\psi^{-1}\left(X - \psi(a)\right) = r_\psi^{-1}X,$$

which, together with (2.4), gives

$$\begin{aligned} q(X) + \alpha r_\psi \Pi(\omega) &= \alpha r_\psi \sum_{l=0}^{k} \frac{\varphi_v^{(l)}(a)}{l!}\left(\psi^{-1}(X) - a\right)^l \\ &= \alpha r_\psi \sum_{l=0}^{k} \frac{\varphi_v^{(l)}(a)}{l!} r_\psi^{-l} X^l. \end{aligned} \tag{6.18}$$

Thus,

$$\|q(X)\|_2^2 = \alpha^2 r_\psi^2 \left|\varphi_v(a) - \Pi(\omega)\right|^2 + \sum_{l=1}^{k}\left|\alpha r_\psi \frac{\varphi_v^{(l)}(a)}{l!} r_\psi^{-l}\right|^2. \tag{6.19}$$

By Lemma 2.4, since $v \in \mathcal{U}_{n,j}$, and from (6.16),

$$\left|\varphi_v^{(l)}(a)\right| \le l!C^l\left|\varphi_v'(0)\right| \le l!C^l 2\alpha^{-1} \text{ for } 1 \le l \le k. \tag{6.20}$$

Since $\omega \in E$, we have $\mathrm{supp}(\beta_\omega) \subset \Pi^{-1}\left(\Pi\omega\right)$. From this and $\beta_\omega\left([vu]\right) > 0$, it follows that there exists $\omega' \in [vu]$ such that $\Pi(\omega) = \Pi(\omega') = \varphi_{vu}\Pi(\sigma^{n+n'}(\omega'))$. Additionally, from $a = \varphi_u(0)$, we get $\varphi_v(a) = \varphi_{vu}(0)$. Hence, by Lemma 2.3, by bounded distortion, since $\Pi(\sigma^{n+n'}(\omega')) \in \Omega$, and from (6.15) and (6.16),

$$\begin{aligned} \left|\varphi_v(a) - \Pi(\omega)\right| &= \left|\varphi_{vu}(0) - \varphi_{vu}\Pi(\sigma^{n+n'}(\omega'))\right| \\ &\le C\left|\varphi_v'(0)\right|\left|\varphi_u'(0)\right| \le 2C^2\alpha^{-1}r_\psi^{-1}. \end{aligned}$$

From this, (6.19) and (6.20),

$$\|q(X)\|_2^2 \le 4C^4 + \sum_{l=1}^{k} 4C^{2l} r_\psi^{2-2l} \le 8kC^{2k}.$$

Let $z \in \mathbb{D}$ be given. From (6.18),

$$q'(z) = \alpha r_\psi \sum_{l=1}^{k} \frac{\varphi_v^{(l)}(a)}{(l-1)!} r_\psi^{-l} z^{l-1}.$$

From (6.15) and (6.20), it follows that for $2 \le l \le k$

$$\left| \alpha \frac{\varphi_v^{(l)}(a)}{(l-1)!} r_\psi^{1-l} z^{l-1} \right| \le 2kC^{2k-1} \cdot 2^{-n'(\chi-\delta)}.$$

Moreover, by (6.16) and bounded distortion, we get $|\alpha \varphi_v'(a)| \ge \frac{1}{2}C^{-1}$. Since $C, k \ll n'$ and we can assume that $\delta < \chi/2$, all of this implies that $|q'(z)| \ge \frac{1}{4}C^{-1}$. We have thus shown that,

$$\|q\|_2 \le \left(8kC^{2k}\right)^{1/2} \text{ and } |q'(z)| \ge \frac{1}{4}C^{-1} \text{ for } q \in \operatorname{supp}(\nu) \text{ and } z \in \mathbb{D}. \tag{6.21}$$

Let $v_1, v_2 \in \mathcal{U}_{n,j}$ be distinct, and for $i = 1, 2$ set $q_i := S_{\alpha r_\psi} T_{-\Pi(\omega)} R_{\psi^{-1}} P_{k,a} \varphi_{v_i}$. Assume by contradiction that $\mathcal{D}_{Mn}^{\mathcal{P}_k}(q_1) = \mathcal{D}_{Mn}^{\mathcal{P}_k}(q_2)$. By the definition of $\mathcal{D}_{Mn}^{\mathcal{P}_k}$ (see Section 2.7), this implies that

$$|q_1(z) - q_2(z)| \le (k+1)2^{1-Mn} \text{ for } z \in \mathbb{D}. \tag{6.22}$$

On the other hand, by (6.3), we have $\|\varphi_{v_1 u} - \varphi_{v_2 u}\|_\Omega \ge c^{n+n'}$. Thus, since $\psi \circ \varphi_u(\Omega) \subset \mathbb{D}$, there exists $z_0 \in \mathbb{D}$ such that $\psi^{-1}(z_0) \in \varphi_u(\Omega)$ and

$$\left|\varphi_{v_1}\left(\psi^{-1}(z_0)\right) - \varphi_{v_2}\left(\psi^{-1}(z_0)\right)\right| \ge c^{n+n'}.$$

Additionally, from $\psi^{-1}(z_0) \in \varphi_u(\Omega)$ and (6.12),

$$\left|\left(\varphi_{v_i} - P_{k,a}\varphi_{v_i}\right)\left(\psi^{-1}(z_0)\right)\right| \le 2^{-Mn-\chi(n+n')} \text{ for } i = 1, 2.$$

From the last two inequalities and since $c^{-1} \ll M$,

$$\left|P_{k,a}\varphi_{v_1}\left(\psi^{-1}(z_0)\right) - P_{k,a}\varphi_{v_2}\left(\psi^{-1}(z_0)\right)\right| \ge c^{n+n'}/2,$$

which gives $|q_1(z_0) - q_2(z_0)| \ge c^{n+n'}/2$. Since $c^{-1} \ll M$ and $k \ll n$, this contradicts (6.22). Hence we must have $\mathcal{D}_{Mn}^{\mathcal{P}_k}(q_1) \ne \mathcal{D}_{Mn}^{\mathcal{P}_k}(q_2)$, which implies

$$\frac{1}{Mn} H\left(\nu, \mathcal{D}_{Mn}^{\mathcal{P}_k}\right) = \frac{1}{Mn} H\left(\left(\beta_\omega\right)_{F_{j,u}}, \mathcal{C}_n\right) \ge \frac{\Delta}{6M}, \tag{6.23}$$

where the last inequality follows from $(j, u) \in \mathcal{Q}$.

For $z \in \Omega_0$ we have $(\psi \circ \varphi_u)'(z) = r_\psi \varphi_u'(z)$. Thus, by Lemma 2.2 and (6.15),

$$C^{-2} \le |(\psi \circ \varphi_u)'(z)| \le C^2,$$

which implies that $\psi \circ \varphi_u \in \mathcal{F}_{1/C^2}$. Also note that $\psi \circ \varphi_u(\Omega) \subset \mathbb{D}$. From these facts, from (6.21) and (6.23), since $\dim \mu < 2$, by the relations (6.2), and by Theorem 5.1,

$$\frac{1}{Mn} H\left(\nu.\left(\psi\varphi_u\mu\right), \mathcal{D}_{Mn}\right) \ge \dim \mu + \rho.$$

Hence, by (6.17),

$$g(\omega, j, u) \ge \dim \mu + \rho - O(\delta) \text{ for all } (j, u) \in \mathcal{Q}. \tag{6.24}$$

We can now complete the proof. From (6.7), (6.11), and (6.24),

$$\begin{aligned} g(\omega) &\geq \sum_{(j,u)\in\mathcal{Q}} \beta_\omega(F_{j,u})(\dim\mu+\rho-O(\delta)) \\ &+ \sum_{(j,u)\in(J\times\mathcal{U}_{n'})\setminus\mathcal{Q}} \beta_\omega(F_{j,u})(\dim\mu-O(\delta)). \end{aligned}$$

Thus, by (6.10) and since $\cup_{(j,u)\in J\times\mathcal{U}_{n'}} F_{j,u} = [\mathcal{U}_n]\cap\sigma^{-n}[\mathcal{U}_{n'}]$,

$$g(\omega) \geq \frac{\rho\Delta}{6\log|\Lambda|} + \beta_\omega\left([\mathcal{U}_n]\cap\sigma^{-n}[\mathcal{U}_{n'}]\right)(\dim\mu-O(\delta)),$$

which holds for all $\omega\in E$. From this and by the definition of E,

$$g(\omega) \geq \frac{\rho\Delta}{6\log|\Lambda|} + \dim\mu - O(\delta) \text{ for all } \omega\in E.$$

Hence, from (6.5) and since $\beta(E) > 1-\delta$,

$$\dim\mu + 2\delta \geq (1-\delta)\left(\frac{\rho\Delta}{6\log|\Lambda|} + \dim\mu - O(\delta)\right).$$

But since $\Delta^{-1}, \rho^{-1} \ll \delta^{-1}$, this yields the desired contradiction, completing the proof of the theorem. $\square$

7. Proof of Corollary 1.6

In this section we prove our result concerning the dimension of self-conformal sets, Corollary 1.6. The proof requires some preparation.

Lemma 7.1. *Let $n\geq 1$ and let $\psi_1,\psi_2:\Omega\to\mathbb{C}$ be holomorphic and injective. Suppose that $\psi_j\circ\varphi_u\circ\psi_j^{-1}:\psi_j(\Omega)\to\mathbb{C}$ is homothetic for each $j=1,2$ and $u\in\Lambda^n$. Then there exists $0\neq w\in\mathbb{C}$ such that $\psi_2'(z)\mathbb{R}=\psi_1'(z)w\mathbb{R}$ for all $z\in K_\Phi$.*

Proof. For $j=1,2$ and $u\in\Lambda^n$ set $h_{j,u}:=\psi_j\circ\varphi_u\circ\psi_j^{-1}$. Let $z\in K_\Phi$ and $\omega\in\Lambda^{\mathbb{N}}$ be with $\Pi\omega=z$, and for $k\geq 0$ set $u_k:=(\sigma^{kn}\omega)|_n$. For $j=1,2$ and $m\geq 1$,

$$\psi_j^{-1}\circ h_{j,u_0}\circ\ldots\circ h_{j,u_{m-1}}\circ\psi_j = \varphi_{u_0}\circ\ldots\circ\varphi_{u_{m-1}} = \varphi_{\omega|_{mn}}.$$

Thus, since the maps $h_{j,u}$ are homothetic,

$$\begin{aligned} \left(\psi_j^{-1}\circ h_{j,u_0}\circ\ldots\circ h_{j,u_{m-1}}\circ\psi_j\right)'(0)\mathbb{R} &= \left(\psi_j^{-1}\right)'\left(h_{j,u_0}\circ\ldots\circ h_{j,u_{m-1}}\circ\psi_j(0)\right)\psi_j'(0)\mathbb{R} \\ &= \left(\psi_j^{-1}\right)'\left(\psi_j\circ\varphi_{\omega|_{mn}}(0)\right)\psi_j'(0)\mathbb{R} = \left(\psi_j'\left(\varphi_{\omega|_{mn}}(0)\right)\right)^{-1}\psi_j'(0)\mathbb{R}. \end{aligned}$$

From the last two equalities,

$$\left(\psi_1'\left(\varphi_{\omega|_{mn}}(0)\right)\right)^{-1}\psi_1'(0)\mathbb{R} = \left(\psi_2'\left(\varphi_{\omega|_{mn}}(0)\right)\right)^{-1}\psi_2'(0)\mathbb{R}.$$

Hence, by letting m tend to ∞,

$$\left(\psi_1'(z)\right)^{-1}\psi_1'(0)\mathbb{R} = \left(\psi_2'(z)\right)^{-1}\psi_2'(0)\mathbb{R},$$

which completes the proof of the lemma with $w=\psi_2'(0)/\psi_1'(0)$. $\square$

Lemma 7.2. *Let $n\geq 1$ be given. Then at least one of the IFSs $\{\varphi_u\}_{u\in\Lambda^n}$ and $\{\varphi_u\}_{u\in\Lambda^{n+1}}$ is not holomorphically conjugate to a homothetic IFS.*

Proof. Assume by contradiction that the lemma is false. Then there exist holomorphic and injective functions $\psi_1, \psi_2 : \Omega \to \mathbb{C}$ such that $\psi_1 \circ \varphi_u \circ \psi_1^{-1}$ is homothetic for each $u \in \Lambda^n$ and $\psi_2 \circ \varphi_u \circ \psi_2^{-1}$ is homothetic for each $u \in \Lambda^{n+1}$. It is clear that $\psi_j \circ \varphi_u \circ \psi_j^{-1}$ is homothetic for each $j = 1, 2$ and $u \in \Lambda^{n(n+1)}$. Hence, by Lemma 7.1, there exists $0 \neq w \in \mathbb{C}$ such that $\psi_2'(z)\mathbb{R} = \psi_1'(z)w\mathbb{R}$ for all $z \in K_\Phi$. This also implies that for each $z \in K_\Phi$

$$\left(\psi_2^{-1}\right)'(\psi_2(z))\mathbb{R} = \frac{1}{\psi_2'(z)}\mathbb{R} = \frac{1}{\psi_1'(z)w}\mathbb{R} = w^{-1}\left(\psi_1^{-1}\right)'(\psi_1(z))\mathbb{R}.$$

Let $i \in \Lambda$ and set $g := \psi_2 \circ \varphi_i \circ \psi_2^{-1}$. Given $u \in \Lambda^n$, it holds that $\psi_2 \circ \varphi_{ui} \circ \psi_2^{-1}$ is homothetic. Hence, for $z \in K_\Phi$

$$\begin{aligned}\mathbb{R} &= \left(\psi_2 \circ \varphi_{ui} \circ \psi_2^{-1}\right)'(\psi_2 z)\mathbb{R} = \left(\psi_2 \circ \varphi_u \circ \psi_2^{-1} \circ \psi_2 \circ \varphi_i \circ \psi_2^{-1}\right)'(\psi_2 z)\mathbb{R} \\ &= \left(\psi_2'\left(\varphi_{ui}(z)\right)\right) \cdot \left(\varphi_u'\left(\varphi_i(z)\right)\right) \cdot \Big(\left(\psi_2^{-1}\right)'\left(\psi_2 \circ \varphi_i(z)\right)\Big) \cdot \left(g'(\psi_2 z)\right)\mathbb{R}.\end{aligned}$$

Thus, since

$$\psi_2'\left(\varphi_{ui}(z)\right)\mathbb{R} = \psi_1'\left(\varphi_{ui}(z)\right)w\mathbb{R}$$

and

$$\left(\psi_2^{-1}\right)'\left(\psi_2 \circ \varphi_i(z)\right)\mathbb{R} = w^{-1}\left(\psi_1^{-1}\right)'\left(\psi_1 \circ \varphi_i(z)\right)\mathbb{R},$$

we obtain that

$$\begin{aligned}\mathbb{R} &= \left(\psi_1'\left(\varphi_{ui}(z)\right)\right) \cdot \left(\varphi_u'\left(\varphi_i(z)\right)\right) \cdot \Big(\left(\psi_1^{-1}\right)'\left(\psi_1 \circ \varphi_i(z)\right)\Big) \cdot \left(g'(\psi_2 z)\right)\mathbb{R} \\ &= \Big(\left(\psi_1 \circ \varphi_u \circ \psi^{-1}\right)'\left(\psi_1 \circ \varphi_i(z)\right)\Big) \cdot \left(g'(\psi_2 z)\right)\mathbb{R} = \left(g'(\psi_2 z)\right)\mathbb{R},\end{aligned}$$

where the last equality holds since $\psi_1 \circ \varphi_u \circ \psi_1^{-1}$ is homothetic.

We have thus shown that $\left(\psi_2 \circ \varphi_i \circ \psi_2^{-1}\right)'(\psi_2 z)\mathbb{R} = \mathbb{R}$ for all $z \in K_\Phi$. Hence, by Lemma 4.7, it follows that $\psi_2 \circ \varphi_i \circ \psi_2^{-1}$ is homothetic. Since this holds for all $i \in \Lambda$, we obtain that Φ is holomorphically conjugate to a homothetic IFS, which contradicts our standing assumption and completes the proof of the lemma. □

Lemma 7.3. *Let $n \geq 1$ be given, set $\Phi_n := \{\varphi_u\}_{u \in \Lambda^n}$, and let $\Gamma \subset \Omega$ be a regular real-analytic curve which is closed in Ω. Then Γ is not Φ_n-invariant.*

Proof. Assume by contradiction that Γ is Φ_n-invariant. It is clear that K_Φ is also the self-conformal set associated to Φ_n. From this, and since Γ is Φ_n-invariant and closed in Ω, it follows easily that $K_\Phi \subset \Gamma$. But since μ is supported on K_Φ, this contradicts Proposition 3.1, which completes the proof of the lemma. □

Proof of Corollary 1.6. Using [5, Theorem 1.22], it can be shown that there exists a unique σ-invariant $\nu \in \mathcal{M}\left(\Lambda^{\mathbb{N}}\right)$ such that $s(\Phi) = h(\nu)/\chi(\Phi, \nu)$. Here $h(\nu)$ is the entropy of ν, and $\chi(\Phi, \nu)$ is the Lyapunov exponent associated to Φ and ν, defined by

$$\chi(\Phi, \nu) := -\int \log\left|\varphi_{\omega_0}'\left(\Pi(\sigma\omega)\right)\right| d\nu(\omega).$$

Moreover, by the construction of ν (again see [5]), we have $\nu\left([u]\right) > 0$ for $u \in \Lambda^n$.

Given $n \geq 1$, set $\Phi_n := \{\varphi_u\}_{u \in \Lambda^n}$ and $p_n := \left(\nu\left([u]\right)\right)_{u \in \Lambda^n}$, and denote by μ_n the self-conformal measure corresponding to Φ_n and p_n. The Lyapunov exponent

$\chi(\Phi_n, p_n)$ associated to Φ_n and p_n is defined as in (2.2). It is easy to verify that $\frac{1}{n}H(p_n) \xrightarrow{n} h(\nu)$ and $\frac{1}{n}\chi(\Phi_n, p_n) \xrightarrow{n} \chi(\Phi, \nu)$, and so

$$\lim_{n\to\infty} (H(p_n)/\chi(\Phi_n, p_n)) = s(\Phi). \tag{7.1}$$

Since Φ is exponentially separated and its maps do not have a common fixed point, it is easy to verify that these two properties also hold for Φ_n for every $n \geq 1$. Thus, by Lemmas 7.2 and 7.3, there exist arbitrarily large $n \geq 1$ for which the conditions of Theorem 1.5 are satisfied with Φ_n in place of Φ. Hence, by that theorem and since μ_n is supported on K_Φ for all $n \geq 1$,

$$\min\{2, H(p_n)/\chi(\Phi_n, p_n)\} = \dim \mu_n \leq \dim_H K_\Phi$$

for arbitrarily large $n \geq 1$. This together with (7.1) shows that $\min\{2, s(\Phi)\} \leq \dim_H K_\Phi$. Since the reverse inequality always holds, this completes the proof of the corollary. $\square$

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