INTERMEDIATE DIMENSIONS UNDER SELF-AFFINE CODINGS

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ABSTRACT. Intermediate dimensions were recently introduced by Falconer, Fraser, and Kempton [Math. Z., 296, (2020)] to interpolate between the Hausdorff and box-counting dimensions. In this paper, we show that for every subset E of the symbolic space, the intermediate dimensions of the projections of E under typical self-affine coding maps are constant and given by formulas in terms of capacities. Moreover, we extend the results to the generalized intermediate dimensions introduced by Banaji [Monatsh. Math., 202, (2023)] in several settings, including the orthogonal projections in Euclidean spaces and the images of fractional Brownian motions.

1. INTRODUCTION

The study on the dimensions of projections of sets has a long history. For a survey of this topic, please refer to [12]. In this paper, we focus on the intermediate dimensions of projections of sets under the coding maps associated with typical affine iterated function systems.

In what follows, we fix a family of $d \times d$ invertible real matrices $T_1, \ldots, T_m$ with $\|T_j\| < 1$ for $1 \leq j \leq m$. Let $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{dm}$. By an *affine iterated function system* (affine IFS) we mean a finite family $\mathcal{F}^{\mathbf{a}} = \{f_j^{\mathbf{a}}\}_{j=1}^m$ of affine maps taking the form

$$f_j^{\mathbf{a}}(x) = T_j x + a_j \quad \text{for } 1 \leq j \leq m.$$

Here we write $f_j^{\mathbf{a}}$ instead of f_j to emphasize its dependence on $\mathbf{a}$. It is well known [17] that there exists a unique non-empty compact set $K^{\mathbf{a}}$ such that

$$K^{\mathbf{a}} = \bigcup_{j=1}^{m} f_j^{\mathbf{a}}(K^{\mathbf{a}}).$$

We call $K^{\mathbf{a}}$ the *self-affine set* generated by $\mathcal{F}^{\mathbf{a}}$. Write $\Sigma := \{1, \ldots, m\}^{\mathbb{N}}$. The (self-affine) *coding map* $\pi^{\mathbf{a}} \colon \Sigma \to \mathbb{R}^d$ associated with $\mathcal{F}^{\mathbf{a}}$ is

$$\pi^{\mathbf{a}}(\mathbf{i}) := \lim_{n \to \infty} f_{i_1}^{\mathbf{a}} \circ \cdots \circ f_{i_n}^{\mathbf{a}}(0) \quad \text{for } \mathbf{i} = i_1 \ldots i_n \ldots \in \Sigma. \tag{1.1}$$

It is well known [17] that $K^{\mathbf{a}} = \pi^{\mathbf{a}}(\Sigma)$.

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There is a good deal of work studying various dimensional properties of projected sets and measures under typical coding maps [7, 14, 18, 19, 20, 21, 25]. Let $\mathcal{L}^d$ denote the Lebesgue measure on $\mathbb{R}^d$. In his seminal paper [7], Falconer showed that the Hausdorff and box-counting dimensions of self-affine sets $K^{\mathbf{a}} = \pi^{\mathbf{a}}(\Sigma)$ remain as a common constant for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$ provided that $\|T_j\| < 1/3$ for all j. The upper bound in this norm condition was later relaxed to $1/2$ by Solomyak [25]. Assuming $\|T_j\| < 1/2$ for all j, very recently Feng, Lo, and Ma [14] showed that for every Borel set $E \subset \Sigma$, each of the Hausdorff, packing, upper, and lower box-counting dimensions of $\pi^{\mathbf{a}}(E)$ is constant for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$. In this paper, letting $E \subset \Sigma$, we obtain an analogous constancy result about the intermediate dimensions of $\pi^{\mathbf{a}}(E)$ for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

Intermediate dimensions were introduced by Falconer, Fraser, and Kempton [13] to interpolate between the Hausdorff and box-counting dimensions; see [11] for a survey. To avoid problems of definition, throughout the paper we assume that all the sets, whose dimensions are considered, are non-empty and bounded. Denote the diameter of a set $U \subset \mathbb{R}^d$ by $|U|$.

Definition 1.1. Let $F \subset \mathbb{R}^d$. For $0 \le \theta \le 1$, the *upper θ-intermediate dimension* of F is defined by

$$\begin{aligned}\overline{\dim}_\theta F = \inf\{s \ge 0\colon\ &\text{for all } \varepsilon > 0, \text{ there exists } r_0 \in (0,1] \text{ such that for all } r \in (0, r_0),\\ &\text{there exists a cover } \{U_i\} \text{ of } F \text{ such that } r^{1/\theta} \le |U_i| \le r \text{ for all } i \text{ and } \sum_i |U_i|^s \le \varepsilon\}\end{aligned}$$

and the *lower θ-intermediate dimension* of F is defined by

$$\begin{aligned}\underline{\dim}_\theta F = \inf\{s \ge 0\colon\ &\text{for all } \varepsilon > 0 \text{ and } r_0 \in (0,1], \text{ there exists } r \in (0, r_0) \text{ and}\\ &\text{a cover } \{U_i\} \text{ of } F \text{ such that } r^{1/\theta} \le |U_i| \le r \text{ for all } i \text{ and } \sum_i |U_i|^s \le \varepsilon\}.\end{aligned}$$

It is immediate that the Hausdorff dimension $\dim_{\mathrm{H}} F$, the upper box-counting dimension $\overline{\dim}_{\mathrm{B}} F$, and the lower box-counting dimension $\underline{\dim}_{\mathrm{B}} F$ are the extreme cases of the θ-intermediate dimensions. Specifically,

$$\dim_{\mathrm{H}} F = \underline{\dim}_0 F = \overline{\dim}_0 F, \quad \overline{\dim}_{\mathrm{B}} F = \overline{\dim}_1 F, \quad \text{and} \quad \underline{\dim}_{\mathrm{B}} F = \underline{\dim}_1 F.$$

Despite their extremely recent introduction, the intermediate dimensions have already seen interesting applications. For example, Burrell, Falconer and Fraser [6, Section 6] showed that if F is a subset of $\mathbb{R}^d$ such that $\lim_{\theta \to 0} \underline{\dim}_\theta F = \dim_{\mathrm{H}} F$, then $\dim_{\mathrm{H}} F \ge m$ if and only if $\underline{\dim}_{\mathrm{B}} P_V F = m$ for almost all m-dimensional subspace V of $\mathbb{R}^d$, where P_V denotes the orthogonal projection onto V. More recently Banaji and Kolossváry [2] determined a precise formula for the intermediate dimensions of Bedford-McMullen carpets and made an unexpected connection to multifractal analysis.

Below we state our first main result on the θ-intermediate dimensions of $\pi^{\mathbf{a}}(E)$ for $E \subset \Sigma$ in terms of the *capacity dimensions* $\underline{\dim}_{\mathrm{C},\theta} E$, $\overline{\dim}_{\mathrm{C},\theta} E$ whose rigorous definitions are given in Definition 2.4.

Theorem 1.2. *Let $0 < \theta \le 1$ and $E \subset \Sigma$. Then the following hold.*

(i) *For all* $\mathbf{a} \in \mathbb{R}^{md}$,

$$\underline{\dim}_\theta \pi^{\mathbf{a}}(E) \le \underline{\dim}_{C,\theta} E \quad \text{and} \quad \overline{\dim}_\theta \pi^{\mathbf{a}}(E) \le \overline{\dim}_{C,\theta} E.$$

(ii) *Assume* $\|T_j\| < 1/2$ *for* $1 \le j \le m$. *Then for* $\mathcal{L}^{dm}$*-a.e.* $\mathbf{a} \in \mathbb{R}^{dm}$,

$$\underline{\dim}_\theta \pi^{\mathbf{a}}(E) = \underline{\dim}_{C,\theta} E \quad \text{and} \quad \overline{\dim}_\theta \pi^{\mathbf{a}}(E) = \overline{\dim}_{C,\theta} E.$$

We remark that the assumption that $\|T_j\| < 1/2$ for $1 \le j \le m$ can be weaken to $\max_{i \ne j}(\|T_i\| + \|T_j\|) < 1$. Indeed the first assumption is only used to guarantee the self-affine transversality; see Lemma 4.2. As pointed out in [4, Proposition 9.4.1], the second assumption is sufficient for the self-affine transversality.

Theorem 1.2 is proved through a capacity approach by adapting and extending some ideas in [6, 10, 14]. Our definitions of kernels are inspired by, but different from that of Burrell, Falconer, and Fraser [6] where the projection theorems are established for the θ-intermediate dimensions under the orthogonal projections in Euclidean spaces. It is these new kernels that reveal a unified computational scheme and pave the way for the extensions to the generalized intermediate dimensions.

In [1], Banaji generalized the θ-intermediate dimensions to the so-called Φ*-intermediate dimensions* $\underline{\dim}_\Phi F$, $\overline{\dim}_\Phi F$ (see Definition 5.1) by replacing the size condition $r^{1/\theta} \le |U_i| \le r$ in Definition 1.1 with $\Phi(r) \le |U_i| \le r$, where Φ is an admissible function. Here a function Φ is called *admissible* if there exists some $Y > 0$ such that Φ is monotonic on $(0, Y)$, and satisfies $0 < \Phi(r) \le r$ for $0 < r < Y$ and $\lim_{r\to 0} \Phi(r)/r = 0$. In particular, we get the θ-intermediate dimensions when $\Phi(r) = r^{1/\theta}$ $(0 < \theta < 1)$ and the box-counting dimensions when $\Phi(r) = -r/\log r$ (see [1, Proposition 3.2]).

It is natural to ask whether there are some results analogous to Theorem 1.2 for the Φ-intermediate dimensions. Our answer is affirmative. Moreover, our strategy can be exploited to study the Φ-intermediate dimensions in several settings, including the orthogonal projections in Euclidean spaces and the images of fractional Brownian motions. For the clarity of illustration, we separately state the settings where we study the Φ-intermediate dimensions.

Setting 1.3. Let $T_1, \ldots, T_m$ be a fixed family of contracting $d \times d$ invertible real matrices. Write $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{dm}$, let $\pi^{\mathbf{a}} \colon \Sigma \to \mathbb{R}^d$ be the coding map associated with the affine IFS $\{T_j x + a_j\}_{j=1}^m$ (see (1.1)).

Setting 1.4. Let $G(d, m)$ be the Grassmannian of m-dimensional subspaces of $\mathbb{R}^d$ and $\gamma_{d,m}$ be the natural invariant probability measure on $G(d, m)$. For $V \in G(d, m)$, let P_V be the orthogonal projection from $\mathbb{R}^d$ onto V.

Setting 1.5. For $0 < \alpha < 1$, the *index-*α *fractional Brownian motion* is the Gaussian random function $B_\alpha \colon \mathbb{R}^d \to \mathbb{R}^m$ that with probability 1 is continuous with $B_\alpha(0) = 0$ and such that the increments $B_\alpha(x) - B_\alpha(y)$ are multivariate normal with the mean vector $0 \in \mathbb{R}^m$ and the covariance matrix $\operatorname{diag}(|x-y|^{2\alpha}, \ldots, |x-y|^{2\alpha}) \in \mathbb{R}^{m\times m}$. Denote the underlying probability space as $(\Omega, \mathbb{P})$. In particular, $B_\alpha = (B_{\alpha,1}, \ldots, B_{\alpha,m})$, where $B_{\alpha,i} \colon \mathbb{R}^d \to \mathbb{R}$ are independent index-α fractional Brownian motions with distributions given by

$$\mathbb{P}\{B_{\alpha,i}(x) - B_{\alpha,i}(y) \in A\} = \frac{1}{\sqrt{2\pi}|x-y|^{\alpha}} \int_A \exp\left(-\frac{t^2}{2|x-y|^{2\alpha}}\right) dt \tag{1.2}$$

for each Borel set $A \subset \mathbb{R}$.

Now we are ready to present our results for the Φ-intermediate dimensions using the *generalized capacity dimensions* $\underline{\dim}_{\mathrm{C},\Phi} E$, $\overline{\dim}_{\mathrm{C},\Phi} E$ (see Definition 5.6) and *generalized dimension profiles* $\underline{\dim}^{\tau}_{\Phi} E$, $\underline{\dim}^{\tau}_{\Phi} E$ (see Definition 5.7).

Theorem 1.6. *Let Φ be an admissible function. Suppose*

$$\lim_{r\to 0} r^{\varepsilon} \log \Phi(r) = 0 \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

Then the following hold.

(i) In Setting 1.3, let $E \subset \Sigma$. Then for all $\mathbf{a} \in \mathbb{R}^{dm}$,

$$\underline{\dim}_{\Phi} \pi^{\mathbf{a}}(E) \le \underline{\dim}_{\mathrm{C},\Phi} E \quad \text{and} \quad \overline{\dim}_{\Phi} \pi^{\mathbf{a}}(E) \le \overline{\dim}_{\mathrm{C},\Phi} E.$$

Assume $\|T_j\| < 1/2$ for $1 \le j \le m$. Then for $\mathcal{L}^{dm}$-a.e. $\mathbf{a} \in \mathbb{R}^{dm}$,

$$\underline{\dim}_{\Phi} \pi^{\mathbf{a}}(E) = \underline{\dim}_{\mathrm{C},\Phi} E \quad \text{and} \quad \overline{\dim}_{\Phi} \pi^{\mathbf{a}}(E) = \overline{\dim}_{\mathrm{C},\Phi} E.$$

(ii) In Setting 1.4, let $E \subset \mathbb{R}^d$. Then for all $V \subset G(d,m)$,

$$\underline{\dim}_{\Phi} P_V E \le \underline{\dim}^{m}_{\Phi} E \quad \text{and} \quad \overline{\dim}_{\Phi} P_V E \le \overline{\dim}^{m}_{\Phi} E. \tag{1.4}$$

Moreover, for $\gamma_{d,m}$-a.e. $V \in G(d,m)$,

$$\underline{\dim}_{\Phi} P_V E = \underline{\dim}^{m}_{\Phi} E \quad \text{and} \quad \overline{\dim}_{\Phi} P_V E = \overline{\dim}^{m}_{\Phi} E.$$

(iii) In Setting 1.5, let $E \subset \mathbb{R}^d$. Then almost surely,

$$\underline{\dim}_{\Phi} B_{\alpha}(E) = \frac{1}{\alpha} \underline{\dim}^{\alpha m}_{\Phi_{\alpha}} E \quad \text{and} \quad \overline{\dim}_{\Phi} B_{\alpha}(E) = \frac{1}{\alpha} \overline{\dim}^{\alpha m}_{\Phi_{\alpha}} E, \tag{1.5}$$

where Φ_{α} is defined in (5.12).

In [1], Banaji asks whether the potential-theoretic methods in [5, 6] can be adapted to study the Φ-intermediate dimensions. This is answered affirmatively by Theorem 1.6 based on the kernels in Definition 5.3 and the condition (1.3). Note that (1.3) holds if $\liminf_{r\to 0} \log r / \log \Phi(r) > 0$, which is satisfied by $\Phi(r) = r^{1/\theta}$ $(0 < \theta < 1)$ and $\Phi(r) = -r/\log r$. There are more general functions satisfying (1.3), for example, $\Phi(r) = r^{-\log r}$.

Recently, there have been substantial advancements on giving conditions for the dimensions of self-affine sets and measures to attain the affinity and Lyapunov dimensions; see [3, 16, 24]. We may expect some reasonable conditions on $\mathbf{a}$ and $T_1, \dots, T_m$ such that the intermediate and capacity dimensions coincide in specific settings. It is not hard to see that the equalities in Theorem 1.2(ii) hold when the underlying IFS consists of similarities and satisfies the strong separation condition (see Example 6.1).

The paper is organized as follows. In Section 2, we provide the definitions of the intermediate and capacity dimensions. Then the proofs of (i) and (ii) of Theorem 1.2 are respectively given in Section 3 and Section 4. After introducing the generalized capacity dimensions and the generalized dimension profiles, we prove Theorem 1.6 in Section 5. Finally, a few remarks are given in the last section.

2. Preliminaries

Throughout this paper, we shall mean by $a \lesssim b$ that $a \leq Cb$ for some positive constant C, and write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. If the constant C depends on some parameters, we sometimes write the parameters in the subscript to emphasize the dependency. For example, if $a \leq Cb$ for some constant C depending on parameters α and β, we write $a \lesssim_{\alpha,\beta} b$. We denote the natural logarithm by log and the natural exponential by exp. By $\#$ we denote the cardinality of a finite set. In a metric space, the closed ball centered at x with radius r is denoted by $B(x,r)$, and the closure of a set E is denoted by $\overline{E}$.

2.1. **Intermediate dimensions.** As noted in [6], it is convenient to work with some equivalent definitions of the θ-intermediate dimensions. These definitions are expressed as limits of logarithms of sums over covers. For $s \geq 0$, $0 < \theta \leq 1$, and $E \subset \mathbb{R}^d$, define

$$S^s_{\theta,r}(E) = \inf \left\{ \sum_i |U_i|^s \colon \{U_i\} \text{ is a cover of } E \text{ with } r^{1/\theta} \leq |U_i| \leq r \right\}.$$

Lemma 2.1. *Let $0 < \theta \leq 1$. Then for $E \subset \mathbb{R}^d$,*

$$\underline{\dim}_\theta E = \left(\textit{the unique } s \in [0,d] \textit{ such that } \liminf_{r\to 0} \frac{\log S^s_{\theta,r}(E)}{-\log r} = 0 \right)$$

and

$$\overline{\dim}_\theta E = \left(\textit{the unique } s \in [0,d] \textit{ such that } \limsup_{r\to 0} \frac{\log S^s_{\theta,r}(E)}{-\log r} = 0 \right).$$

Lemma 2.1 is a direct consequence of the following result.

Lemma 2.2 ([6, Lemma 2.1])**.** *Let $0 < \theta \leq 1$ and $E \subset \mathbb{R}^d$. For $0 < r \leq 1$ and $0 \leq t \leq s \leq d$,*

$$-(1/\theta)(s-t) \leq \frac{\log S^s_{\theta,r}(E)}{-\log r} - \frac{\log S^t_{\theta,r}(E)}{-\log r} \leq -(s-t).$$

In particular, there is a unique $s \in [0,d]$ such that $\liminf_{r\to 0} \frac{\log S^s_{\theta,r}(E)}{-\log r} = 0$ and a unique $s \in [0,d]$ such that $\limsup_{r\to 0} \frac{\log S^s_{\theta,r}(E)}{-\log r} = 0$.

2.2. **Capacity dimensions.** Let $T_1, \ldots, T_m$ be a fixed family of contracting $d \times d$ invertible real matrices. Recall $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$. For $n \in \mathbb{N}$, write $\Sigma_n := \{1, \ldots, m\}^n$ and $\Sigma^* := \bigcup_{n=1}^\infty \Sigma_n \bigcup \{\varnothing\}$, where $\varnothing$ denotes the empty word. Write $|I| := n$ for $I \in \Sigma_n$. For $x = (x_k)_{k=1}^\infty \in \Sigma$ and $n \in \mathbb{N}$, denote $x|n := x_1 \cdots x_n$. For $I = i_1 \cdots i_n \in \Sigma_n$, define

$$[I] := \{x \in \Sigma \colon x|n = I\} \quad \text{and} \quad T_I := T_{i_1} \cdots T_{i_n}.$$

By convention we let $x|0 = \varnothing$ and $T_\varnothing$ be the identity map on $\mathbb{R}^d$. Let $x \wedge y$ denote the common initial segment of $x, y \in \Sigma$. Endow Σ with the canonical metric $d(x,y) := \exp(-|x \wedge y|)$ for $x, y \in \Sigma$. By convention we set $\mu(I) := \mu([I])$ for $I \in \Sigma^*$ and any Borel measure μ on Σ. For any $d \times d$ real matrix T, the singular values of T are decreasingly denoted by $\alpha_1(T), \ldots, \alpha_d(T)$.

Following [19], for $r > 0$ we define

$$Z_r(x \wedge y) = \begin{cases} \prod_{k=1}^d \min\{1, \frac{r}{\alpha_k(T_{x\wedge y})}\} & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} \quad \text{for } x, y \in \Sigma. \tag{2.1}$$

For $0 \leq s \leq d$, $0 < \theta \leq 1$, and $r > 0$, we introduce the *kernel*

$$J^s_{\theta,r}(x \wedge y) := \max_{r^{1/\theta} \leq u \leq r} u^{-s} Z_u(x \wedge y) \quad \text{for } x, y \in \Sigma. \tag{2.2}$$

Let $\mathcal{P}(E)$ denote the set of Borel probability measures supported on a compact set E. For compact set $E \subset \Sigma$, the *capacity* of E is defined by

$$C^s_{\theta,r}(E) = \left(\inf_{\mu \in \mathcal{P}(E)} \iint J^s_{\theta,r}(x \wedge y)\, d\mu(x) d\mu(y) \right)^{-1}. \tag{2.3}$$

By convention we set $C^s_{\theta,r}(E) := C^s_{\theta,r}(\overline{E})$ for non-compact subset $E \subset \Sigma$. Thus we can assume, without loss of generality, that the set whose capacities are considered is compact.

The existence of equilibrium measures for kernels and the relationship between the minimal energy and the corresponding potentials is standard in classical potential theory. We state this in a convenient form for the positive symmetric continuous kernels (cf. [15, Theorem 2.4] or [10, Lemma 2.1]).

Lemma 2.3. *Let E be a non-empty compact set in a metric space, and let $K : E \times E \to (0, +\infty)$ be a continuous function such that $K(x, y) = K(y, x)$. Then there is some measure $\mu_0 \in \mathcal{P}(E)$ such that*

$$\iint K(x, y)\, d\mu_0(x) d\mu_0(y) = \frac{1}{C_K(E)},$$

where $C_K(E) = \left(\inf_{\mu \in \mathcal{P}(E)} \iint K(x, y)\, d\mu(x) d\mu(y)\right)^{-1}$. Moreover,

$$\int K(x, y)\, d\mu_0(y) \geq \frac{1}{C_K(E)} \qquad \textit{for all } x \in E, \tag{2.4}$$

with equality for μ_0-a.e. $x \in E$.

A measure μ_0 in Lemma 2.3 is called an *equilibrium measure* for the *kernel* K. Now we are ready to introduce the capacity dimensions.

Definition 2.4 (capacity dimensions)**.** Let $0 < \theta \leq 1$ and $E \subset \Sigma$. The *lower and upper capacity dimensions* of E are respectively defined by

$$\underline{\dim}_{\mathrm{C},\theta} E = \left(\text{the unique } s \in [0, d] \text{ such that } \liminf_{r \to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} = 0 \right),$$

and

$$\overline{\dim}_{\mathrm{C},\theta} E = \left(\text{the unique } s \in [0, d] \text{ such that } \limsup_{r \to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} = 0 \right).$$

Definition 2.4 is justified by the following lemma, an analog of [6, Lemma 3.2]. For completeness, we include a detailed proof.

Lemma 2.5. *Let $0 < \theta \le 1$ and $E \subset \Sigma$. Then for $0 < r \le 1$ and $0 \le t \le s \le d$,*

$$-(1/\theta)(s-t) \le \frac{\log C^s_{\theta,r}(E)}{-\log r} - \frac{\log C^t_{\theta,r}(E)}{-\log r} \le -(s-t). \tag{2.5}$$

In particular, there is a unique $s \in [0,d]$ such that $\liminf_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} = 0$ and a unique $s \in [0,d]$ such that $\limsup_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} = 0$.

Proof. Since $0 < r \le 1$ and $0 \le t \le s$, it follows from (2.2) that for $x, y \in \Sigma$,

$$\begin{aligned} r^{(1/\theta)(s-t)} J^s_{\theta,r}(x \wedge y) &= r^{(1/\theta)(s-t)} \max_{r^{1/\theta} \le u \le r} u^{-s} Z_u(x \wedge y) \\ &= r^{(1/\theta)(s-t)} \max_{r^{1/\theta} \le u \le r} u^{-(s-t)} u^{-t} Z_u(x \wedge y) \\ &\le \max_{r^{1/\theta} \le u \le r} u^{-t} Z_u(x \wedge y) && \text{by } u \ge r^{1/\theta} \\ &= J^t_{\theta,r}(x \wedge y) \\ &= \max_{r^{1/\theta} \le u \le r} u^{s-t} u^{-s} Z_u(x \wedge y) \\ &\le r^{s-t} J^s_{\theta,r}(x \wedge y) && \text{by } u \le r. \end{aligned} \tag{2.6}$$

Without loss of generality assume that E is compact. By Lemma 2.3, there exists an equilibrium measure μ_0 on E for the kernel $J^t_{\theta,r}(x \wedge y)$. Then

$$\begin{aligned} r^{(1/\theta)(s-t)} (C^s_{\theta,r}(E))^{-1} &\le r^{(1/\theta)(s-t)} \iint J^s_{\theta,r}(x \wedge y)\, d\mu_0(x) d\mu_0(y) \\ &\le \iint J^t_{\theta,r}(x \wedge y)\, d\mu_0(x) d\mu_0(y) && \text{by (2.6)} \\ &= (C^t_{\theta,r}(E))^{-1}, \end{aligned}$$

and so

$$r^{(1/\theta)(s-t)} C^t_{\theta,r}(E) \le C^s_{\theta,r}(E). \tag{2.7}$$

Similarly,

$$C^s_{\theta,r}(E) \le r^{s-t} C^t_{\theta,r}(E). \tag{2.8}$$

By (2.7) and (2.8), taking logarithms and making a rearrangement give (2.5).

Taking limits of the quotients in (2.5) shows that the functions

$$s \mapsto \liminf_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} \quad \text{and} \quad s \mapsto \limsup_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} \tag{2.9}$$

are strictly decreasing and continuous on $[0,d]$. Since $J^0_{\theta,r}(x \wedge y) = Z_r(x \wedge y) \le 1$, we have $C^0_{\theta,r}(E) \ge 1$. This implies

$$\liminf_{r\to 0} \frac{\log C^0_{\theta,r}(E)}{-\log r} \ge 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\log C^0_{\theta,r}(E)}{-\log r} \ge 0. \tag{2.10}$$

On the other hand, since $r^{-d}Z_r(x\wedge y)=\prod_{k=1}^d \min\{1/r, 1/\alpha_k(T_{x\wedge y})\}\geq 1$ for $0<r\leq 1$, we have

$$J_{\theta,r}^d(x\wedge y)=\max_{r^{1/\theta}\leq u\leq r} u^{-d}Z_u(x\wedge y)\geq 1.$$

Hence $C_{\theta,r}^d(E)\leq 1$, and so

$$\liminf_{r\to 0}\frac{\log C_{\theta,r}^d(E)}{-\log r}\leq 0 \quad \text{and} \quad \limsup_{r\to 0}\frac{\log C_{\theta,r}^d(E)}{-\log r}\leq 0. \tag{2.11}$$

Based on (2.10) and (2.11), the proof is completed by the continuity and strict monotonicity of the functions in (2.9). □

3. Proof of Theorem 1.2(i)

We begin with a simple geometric observation.

Lemma 3.1 ([14, Lemma 3.2]). *Let* $\mathbf{a}\in\mathbb{R}^{dm}$. *Then*

$$N_r\left(\pi^{\mathbf{a}}([I])\right)\lesssim_{d,\mathbf{a}}\frac{1}{Z_r(I)}\quad \text{for } I\in\Sigma^* \text{ and } r>0,$$

where $N_r(F)$ *denotes the minimal number of sets with diameter* r *needed to cover any bounded set* $F\subset\mathbb{R}^d$.

Next we deduce an upper bound on $S_{\theta,r}^s(\pi^{\mathbf{a}}(E))$ from a lower bound on the potentials of a measure with respect to the kernel $J_{\theta,r}^s(x\wedge y)$.

Proposition 3.2. *Let* $0\leq s\leq d$, $0<\theta\leq 1$, *and* $\mathbf{a}\in\mathbb{R}^{dm}$. *Let* $E\subset\Sigma$ *be a non-empty compact set. If for* $0<r\leq 1$ *there exist* $\mu\in\mathcal{P}(E)$ *and* $\gamma>0$ *such that*

$$\int J_{\theta,r}^s(x\wedge y)\,d\mu(y)\geq\gamma\quad \text{for all } x\in E,$$

then for all sufficiently small $r>0$,

$$S_{\theta,r}^s(\pi^{\mathbf{a}}(E))\lesssim_{d,\mathbf{a},\theta}\frac{\log(1/r)}{\gamma}.$$

For the proof of Proposition 3.2, we adapt some ideas from the proof of [6, Lemma 4.4]. The overall strategy is to find a cover consisting of balls of relatively large measure and appropriate diameters. To this end, the authors of [6] replace the balls of large measure but diameters exceeding r^θ with the collections of balls of diameter r^θ; see the discussion about $\mathscr{E}_i$ and $\mathscr{F}_i$ in the proof of [6, Lemma 4.4]. However, instead of only dealing with the oversize balls, here we replace each of the cylinders of large measure with the collections of sets of appropriate diameters based on the kernel $J_{\theta,r}^s(x\wedge y)$; see (3.4) and (3.5).

Proof of Proposition 3.2. Let $x\in E$. Set $\ell(x):=\min\{n\in\mathbb{N}\colon \alpha_1(T_{x|n})\leq r^{1/\theta}\}$. Write $\alpha_+:=\max_{1\leq j\leq m}\|T_j\|$ and $\ell:=\lceil(1/\theta)\log r/\log\alpha_+\rceil$. Then $\ell(x)\leq\ell$ since

$$\alpha_1(T_{x|\ell})=\|T_{x|\ell}\|\leq\|T_{x_1}\|\cdots\|T_{x_\ell}\|\leq\alpha_+^\ell\leq r^{1/\theta}.$$

For $n \geq \ell(x)$ and $u \geq r^{1/\theta}$, it follows from (2.1) that $Z_u(x|n) = 1$, and so

$$J^s_{\theta,r}(x|n) = \max_{r^{1/\theta} \leq u \leq r} u^{-s} = r^{-s/\theta}. \tag{3.1}$$

This implies

$$\begin{aligned}
\gamma &\leq \int J^s_{\theta,r}(x \wedge y)\, d\mu(y) \\
&= \sum_{n=0}^{\ell(x)-1} J^s_{\theta,r}(x|n)\mu\{y \in \Sigma\colon |x \wedge y| = n\} \\
&\qquad + r^{-s/\theta} \left(\sum_{n=\ell(x)}^{\infty} \mu\{y \in \Sigma\colon |x \wedge y| = n\} + \mu(\{x\}) \right) && \text{by (3.1)} \\
&= \sum_{n=0}^{\ell(x)-1} J^s_{\theta,r}(x|n)\, [\mu(x|n) - \mu(x|(n+1))] + r^{-s/\theta}\mu(x|\ell(x)) \\
&\leq \sum_{n=0}^{\ell(x)} J^s_{\theta,r}(x|n)\mu(x|n) && \text{by (3.1).}
\end{aligned}$$

Hence there exists an integer $n(x) \in [0, \ell(x)]$ such that

$$J^s_{\theta,r}(x|n(x))\mu(x|n(x)) \geq \frac{\gamma}{\ell(x)+1} \geq \frac{\gamma}{\ell+1}. \tag{3.2}$$

By (2.2), there exists some $\delta(x) \in [r^{1/\theta}, r]$ such that $J^s_{\theta,r}(x|n(x)) = \delta(x)^{-s} Z_{\delta(x)}(x|n(x))$. Then (3.2) implies

$$\delta(x)^{-s} Z_{\delta(x)}(x|n(x))\mu(x|n(x)) \geq \frac{\gamma}{\ell+1}. \tag{3.3}$$

Since $\{[x|n(x)]\}_{x \in E}$ is a cover of E and $n(x) \leq \ell$, we can find a disjoint subcover Γ by the net structure of Σ. By (3.3), for each $I \in \Gamma$ there exists some $\delta_I \in [r^{1/\theta}, r]$ such that

$$\frac{\delta_I^s}{Z_{\delta_I}(I)} \leq \frac{\ell+1}{\gamma}\mu(I). \tag{3.4}$$

Clearly $\{\pi^{\mathbf{a}}([I])\}_{I \in \Gamma}$ is a cover of $\pi^{\mathbf{a}}(E)$ since Γ covers E. By Lemma 3.1, we can find for each $\pi^{\mathbf{a}}([I])$ a cover $\mathcal{D}_I$ consisting of sets with diameter $\delta_I \in [r^{1/\theta}, r]$ such that

$$\#\mathcal{D}_I \lesssim_{d,\mathbf{a}} \frac{1}{Z_{\delta_I}(I)}. \tag{3.5}$$

Then $\bigcup_{I \in \Gamma} \mathcal{D}_I$ is a cover of $\pi^{\mathbf{a}}(E)$ with sets of diameters in $[r^{1/\theta}, r]$. Finally,

$$\begin{aligned}
S^s_{\theta,r}(\pi^{\mathbf{a}}(E)) \leq \sum_{I \in \Gamma} \sum_{B \in \mathcal{D}_I} |B|^s &= \sum_{I \in \Gamma} \#\mathcal{D}_I \cdot \delta_I^s \\
&\lesssim_{d,\mathbf{a}} \sum_{I \in \Gamma} \frac{\delta_I^s}{Z_{\delta_I}(I)} && \text{by (3.5)} \\
&\leq \sum_{I \in \Gamma} \frac{\ell+1}{\gamma}\mu(I) && \text{by (3.4)}
\end{aligned}$$

$$= \frac{\ell+1}{\gamma} \qquad \text{by } \Gamma \text{ disjoint.}$$

This finishes the proof since $\ell + 1 \lesssim_\theta \log(1/r)$ when r is small. □

Now we are ready to prove (i) of Theorem 1.2.

Proof of Theorem 1.2(i). Since the θ-intermediate dimensions and capacity dimensions of a set remain the same after taking closure, without loss of generality we can assume that E is compact.

Let $0 \le s \le d$. For $0 < r \le 1$, by Lemma 2.3 there exists an equilibrium measure $\mu_r \in \mathcal{P}(E)$ for the kernel $J^s_{\theta,r}(x \wedge y)$ such that

$$\int J^s_{\theta,r}(x \wedge y)\, d\mu_r(y) \ge \frac{1}{C^s_{\theta,r}(E)} \quad \text{for all } x \in E.$$

Applying Proposition 3.2 with $\mu = \mu_r$ gives that for all sufficiently small $r > 0$,

$$S^s_{\theta,r}(\pi^{\mathbf{a}}(E)) \lesssim \log(1/r) \cdot C^s_{\theta,r}(E).$$

By taking logarithms and limits,

$$\liminf_{r\to 0} \frac{\log S^s_{\theta,r}(\pi^{\mathbf{a}}(E))}{-\log r} \le \liminf_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r} \tag{3.6}$$

and

$$\limsup_{r\to 0} \frac{\log S^s_{\theta,r}(\pi^{\mathbf{a}}(E))}{-\log r} \le \limsup_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r}. \tag{3.7}$$

Hence Lemma 2.1 and Definition 2.4 show that

$$\underline{\dim}_\theta \pi^{\mathbf{a}}(E) \le \underline{\dim}_{\mathrm{C},\theta} E \quad \text{and} \quad \overline{\dim}_\theta \pi^{\mathbf{a}}(E) \le \overline{\dim}_{\mathrm{C},\theta} E.$$

This completes the proof of Theorem 1.2(i). □

4. Proof of Theorem 1.2(ii)

We begin with a lemma modified from [6, Lemma 5.4], which allows us to control $S^s_{\theta,r}(\pi^{\mathbf{a}}(E))$ from below using the upper bounds on the potentials with respect to the kernel $\psi^s_{\theta,r}(|x-y|)$. For $0 \le s \le d$, $0 < \theta \le 1$ and $0 < r \le 1$, define

$$\psi^s_{\theta,r}(\Delta) = \begin{cases} r^{-s/\theta} & \text{if } 0 \le \Delta \le r^{1/\theta} \\ \Delta^{-s} & \text{if } r^{1/\theta} < \Delta \le r \\ 0 & \text{if } \Delta > r \end{cases} \quad \text{for } \Delta \ge 0. \tag{4.1}$$

Note that $\psi^s_{\theta,r}(\cdot)$ is non-increasing.

Lemma 4.1. *Let $0 \le s \le d$, $0 < \theta \le 1$, and $0 < r \le 1$. Let $E \subset \mathbb{R}^d$ be a non-empty compact set. If there exist $\mu \in \mathcal{P}(E)$ and a Borel subset $F \subset E$, and $\gamma > 0$ such that*

$$\int \psi^s_{\theta,r}(|x-y|)\, d\mu(y) \le \gamma \quad \textit{for all } x \in F, \tag{4.2}$$

then

$$S^s_{\theta,r}(E) \geq \frac{\mu(F)}{\gamma}.$$

We can view Lemma 4.1 as a potential-theoretic version of the mass distribution principle (see [8]).

Proof of Lemma 4.1. Let $x \in F$ and $r^{1/\theta} \leq \delta \leq r$. It follows from (4.1) that $\psi^s_{\theta,r}(|x-y|) \geq \delta^{-s}$ for $y \in B(x,\delta)$. Then (4.2) implies that

$$\gamma \geq \int \psi^s_{\theta,r}(|x-y|)\, d\mu(y) \geq \int_{B(x,\delta)} \psi^s_{\theta,r}(|x-y|)\, d\mu(y) \geq \frac{\mu(B(x,\delta))}{\delta^s},$$

thus

$$\delta^s \geq \frac{\mu(B(x,\delta))}{\gamma}. \tag{4.3}$$

Let $\{U_i\}$ be a cover of F with $r^{1/\theta} \leq |U_i| \leq r$. Without loss of generality we can pick some $x_i \in F \cap U_i$ for each i. Then $U_i \subset B(x_i, |U_i|)$ for each i. Hence by (4.3),

$$\sum_i |U_i|^s \geq \frac{1}{\gamma} \sum_i \mu(B(x_i, |U_i|)) \geq \frac{\mu(F)}{\gamma}.$$

Taking infima over all such covers gives

$$S^s_{\theta,r}(E) \geq S^s_{\theta,r}(F) \geq \frac{\mu(F)}{\gamma}.$$

This finishes the proof. □

The following lemma is contained in the proof of [19, Lemma 5.1] which verifies the so-called *self-affine transversality* in (4.4).

Lemma 4.2 ([19, Lemma 5.1])**.** *Assume* $\|T_j\| < 1/2$ *for* $1 \leq j \leq m$. *Let* $\rho > 0$. *Then for* $x, y \in \Sigma$ *and* $r > 0$,

$$\mathcal{L}^{dm}\{\mathbf{a} \in B_\rho \colon |\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)| \leq r\} \lesssim_{\rho,d} Z_r(x \wedge y), \tag{4.4}$$

where B_ρ *denotes the closed ball in* $\mathbb{R}^{dm}$ *centered at* 0 *with radius* ρ.

Next we exploit Lemma 4.2 to relate the integral of $\mathbf{a} \mapsto \psi^s_{\theta,r}(|\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|)$ to the kernel $J^s_{\theta,r}(x \wedge y)$.

Proposition 4.3. *Assume* $\|T_j\| < 1/2$ *for* $1 \leq j \leq m$. *Let* $\rho > 0$, $0 \leq s \leq d$ *and* $0 < \theta \leq 1$. *Then for* $x, y \in \Sigma$ *and* $0 < r \leq 1$,

$$\int_{B_\rho} \psi^s_{\theta,r}(|\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|)\, d\mathbf{a} \lesssim_{\rho,d,\theta} \log(1/r) J^s_{\theta,r}(x \wedge y). \tag{4.5}$$

Proof. Write $\mathcal{L} := \mathcal{L}^{dm}$ for short. By Lemma 4.2,

$$\mathcal{L}\{\mathbf{a} \in B_\rho \colon |\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)| \leq r\} \lesssim_{\rho,d} Z_r(x \wedge y). \tag{4.6}$$

Set $\Delta_{\mathbf{a}} := |\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|$. Then

$$
\begin{aligned}
&\int_{B_\rho} \psi^s_{\theta,r}(\Delta_{\mathbf{a}})\, d\mathbf{a} \\
&\qquad = \int_{\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}}\le r^{1/\theta}\}} r^{-s/\theta}\, d\mathbf{a} + \int_{\{\mathbf{a}\in B_\rho\colon r^{1/\theta}<\Delta_{\mathbf{a}}\le r\}} \Delta_{\mathbf{a}}^{-s}\, d\mathbf{a} && \text{by (4.1)} \\
&\qquad = r^{-s/\theta}\mathcal{L}\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}} \le r^{1/\theta}\} + \int_0^\infty \mathcal{L}\{\mathbf{a}\in B_\rho\colon r^{1/\theta} < \Delta_{\mathbf{a}} \le r,\ \Delta_{\mathbf{a}}^{-s} \ge t\}\, dt \\
&\qquad = r^{-s/\theta}\mathcal{L}\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}} \le r^{1/\theta}\} + \int_0^{r^{-s/\theta}} \mathcal{L}\{\mathbf{a}\in B_\rho\colon r^{1/\theta} < \Delta_{\mathbf{a}} \le \min\{r, t^{-1/s}\}\}\, dt \\
&\qquad = \int_0^{r^{-s/\theta}} \mathcal{L}\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}} \le \min\{r, t^{-1/s}\}\}\, dt \\
&\qquad = \int_0^{r^{-s}} \mathcal{L}\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}} \le r\}\, dt + \int_{r^{-s}}^{r^{-s/\theta}} \mathcal{L}\{\mathbf{a}\in B_\rho\colon \Delta_{\mathbf{a}} \le t^{-1/s}\}\, dt \\
&\qquad \lesssim_{d,\rho} r^{-s} Z_r(x\wedge y) + \int_{r^{-s}}^{r^{-s/\theta}} Z_{t^{-1/s}}(x\wedge y)\, dt && \text{by (4.6)} \\
&\qquad = r^{-s} Z_r(x\wedge y) + s\int_{r^{1/\theta}}^{r} u^{-(s+1)} Z_u(x\wedge y)\, du && \text{by taking } u = t^{-1/s} \\
&\qquad \le \left(1 + s\int_{r^{1/\theta}}^{r} u^{-1}\, du\right) J^s_{\theta,r}(x\wedge y) && \text{by (2.2)} \\
&\qquad \lesssim_{d,\theta} \log(1/r) J^s_{\theta,r}(x\wedge y),
\end{aligned}
$$

where the last inequality is by $s\int_{r^{1/\theta}}^{r} u^{-1}\, du = s(1/\theta - 1)\log(1/r) \lesssim_{d,\theta} \log(1/r)$. □

Now we are ready to prove Theorem 1.2(ii).

Proof of Theorem 1.2(ii). Our arguments are mainly adapted from the proof of [6, Theorem 5.1]. We focus on the case of the upper θ-intermediate dimensions while the proof for the lower θ-intermediate dimensions is similar. By Theorem 1.2(i), it suffices to prove

$$\overline{\dim}_\theta \pi^{\mathbf{a}}(E) \ge \overline{\dim}_{\mathrm{C},\theta} E$$

for $\mathcal{L}^{dm}$-a.e. $\mathbf{a} \in B_\rho$ and $\rho > 0$.

Let $0 \le s \le d$. Take a sequence $(r_k)_{k=1}^\infty$ tending to 0 such that $0 < r_k \le 2^{-k}$ and

$$\limsup_{k\to\infty} \frac{\log C^s_{\theta,r_k}(E)}{-\log r_k} = \limsup_{r\to 0} \frac{\log C^s_{\theta,r}(E)}{-\log r}. \tag{4.7}$$

By Lemma 2.3, for each $k \in \mathbb{N}$ there is an equilibrium measure μ_k on E for the kernel $J^s_{\theta,r_k}(x\wedge y)$. Write

$$\gamma_k := \frac{1}{C^s_{\theta,r_k}(E)} = \iint J^s_{\theta,r_k}(x\wedge y)\, d\mu_k(x) d\mu_k(y).$$

Let $\rho > 0$. Proposition 4.3 implies that

$$\iint \int_{B_\rho} \psi^s_{\theta,r_k}(|\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|)\, d\mathbf{a}\, d\mu_k(x) d\mu_k(y) \lesssim \gamma_k \log(1/r_k). \tag{4.8}$$

Let $\varepsilon > 0$. Note that there is some $A > 0$ such that $r^{\varepsilon/2}\log(1/r) \le A$ for all $r > 0$. Then summing (4.8) over $k \in \mathbb{N}$ and using Fubini's theorem lead to

$$\int_{B_\rho} \sum_{k=1}^{\infty} \left(r_k^{\varepsilon} \gamma_k^{-1} \iint \psi_{\theta,r_k}^s(|\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|)\, d\mu_k(x) d\mu_k(y) \right) d\mathbf{a}$$
$$\lesssim \sum_{k=1}^{\infty} \log(1/r_k) r_k^{\varepsilon} \le A \sum_{k=1}^{\infty} r_k^{\varepsilon/2} \le A \sum_{k=1}^{\infty} 2^{-k\varepsilon/2} < \infty.$$

Hence for $\mathcal{L}^{dm}$-a.e. $\mathbf{a} \in B_\rho$, there exists $M_{\mathbf{a}} > 0$ such that

$$\iint \psi_{\theta,r_k}^s(|u-v|)\, d\pi^{\mathbf{a}}\mu_k(v) d\pi^{\mathbf{a}}\mu_k(u) \le M_{\mathbf{a}} \gamma_k r_k^{-\varepsilon} \quad \text{for all } k \in \mathbb{N}.$$

Then for each k there exists some Borel $F_k \subset \pi^{\mathbf{a}}(E)$ such that $(\pi^{\mathbf{a}}\mu_k)(F_k) \ge 1/2$ and

$$\int \psi_{\theta,r_k}^s(|u-v|)\, d\pi^{\mathbf{a}}\mu_k(v) \le 2M_{\mathbf{a}} \gamma_k r_k^{-\varepsilon} \quad \text{for all } u \in F_k.$$

Lemma 4.1 implies that

$$S_{\theta,r_k}^s(\pi^{\mathbf{a}}(E)) \ge \frac{1}{2}(2M_{\mathbf{a}}\gamma_k r_k^{-\varepsilon})^{-1} \gtrsim_{\mathbf{a}} r_k^{\varepsilon}\gamma_k^{-1} = r_k^{\varepsilon} C_{\theta,r_k}^s(E),$$

thus

$$\begin{aligned}
\limsup_{k\to\infty} \frac{\log S_{\theta,r_k}^s(\pi^{\mathbf{a}}(E))}{-\log r_k} &\ge \limsup_{k\to\infty} \frac{\log\left(r_k^{\varepsilon} C_{\theta,r_k}^s(E)\right)}{-\log r_k} \\
&= -\varepsilon + \limsup_{k\to\infty} \frac{\log C_{\theta,r_k}^s(E)}{-\log r_k} \\
&= -\varepsilon + \limsup_{r\to 0} \frac{\log C_{\theta,r}^s(E)}{-\log r} \qquad \text{by (4.7)}.
\end{aligned}$$

Letting $\varepsilon \to 0$ gives

$$\limsup_{r\to 0} \frac{\log S_{\theta,r}^s(\pi^{\mathbf{a}}(E))}{-\log r} \ge \limsup_{r\to 0} \frac{\log C_{\theta,r}^s(E)}{-\log r} \quad \text{for } 0 \le s \le d.$$

Finally Lemma 2.1 and Definition 2.4 show that

$$\overline{\dim}_\theta \pi^{\mathbf{a}}(E) \ge \overline{\dim}_{\mathrm{C},\theta} E \quad \text{for } \mathcal{L}^{dm}\text{-a.e. } \mathbf{a} \in B_\rho.$$

The proof for the lower θ-intermediate dimensions is similar. □

5. Generalized intermediate dimensions

In this section, we will prove Theorem 1.6 through a similar strategy of Theorem 1.2. In what follows, let $\Phi\colon (0,Y) \to (0,\infty)$ be an admissible function for some $Y > 0$.

5.1. **Generalized intermediate dimensions.** Following [1], we introduce the generalized intermediate dimensions called the Φ*-intermediate dimensions*.

Definition 5.1 (Φ-intermediate dimensions)**.** For $E \subset \mathbb{R}^d$, its *upper* Φ*-intermediate dimension* is defined by

$$\overline{\dim}_\Phi E = \inf\{s \geq 0\colon \text{ for all } \varepsilon > 0 \text{ there exists } r_0 \in (0,1] \text{ such that for all } r \in (0,r_0), \text{ there exists a cover } \{U_i\} \text{ of } E \text{ such that } \Phi(r) \leq |U_i| \leq r \text{ for all } i \text{ and } \sum_i |U_i|^s \leq \varepsilon\}$$

and its *lower* Φ*-intermediate dimension* is defined by

$$\underline{\dim}_\Phi E = \inf\{s \geq 0\colon \text{ for all } \varepsilon > 0 \text{ and } r_0 \in (0,1] \text{ there exists } r \in (0,r_0) \text{ and a cover } \{U_i\} \text{ of } E \text{ such that } \Phi(r) \leq |U_i| \leq r \text{ for all } i \text{ and } \sum_i |U_i|^s \leq \varepsilon\}.$$

We can describe the Φ-intermediate dimensions by employing a similar approach to that used in defining the Hausdorff dimension with the aid of the Hausdorff measures. For $s \geq 0$, $r > 0$, and $E \subset \mathbb{R}^d$, define

$$S^s_{\Phi,r}(E) = \inf\left\{\sum_i |U_i|^s\colon \{U_i\} \text{ is a cover of } E \text{ with } \Phi(r) \leq |U_i| \leq r\right\}. \tag{5.1}$$

Lemma 5.2. *Let* Φ *be an admissible function and* $E \subset \mathbb{R}^d$*. Then*

$$\overline{\dim}_\Phi E = \inf\{s \geq 0\colon \limsup_{r\to 0} S^s_{\Phi,r}(E) < \infty\} = \inf\{s \geq 0\colon \limsup_{r\to 0} S^s_{\Phi,r}(E) = 0\}$$
$$= \sup\{s \geq 0\colon \limsup_{r\to 0} S^s_{\Phi,r}(E) = \infty\} = \sup\{s \geq 0\colon \limsup_{r\to 0} S^s_{\Phi,r}(E) > 0\}$$

and

$$\underline{\dim}_\Phi E = \inf\{s \geq 0\colon \liminf_{r\to 0} S^s_{\Phi,r}(E) < \infty\} = \inf\{s \geq 0\colon \liminf_{r\to 0} S^s_{\Phi,r}(E) = 0\}$$
$$= \sup\{s \geq 0\colon \liminf_{r\to 0} S^s_{\Phi,r}(E) = \infty\} = \sup\{s \geq 0\colon \liminf_{r\to 0} S^s_{\Phi,r}(E) > 0\}.$$

Proof. Since E is non-empty, we have $S^0_{\Phi,r}(E) \geq 1$. Pick any $x \in E$, then $E \subset B(x,|E|)$. Since E is bounded, we have for $r \leq |E|$,

$$S^d_{\Phi,r}(E) \leq S^d_{\Phi,r}(B(x,|E|)) \leq r^d N_r(B(x,|E|)) \lesssim_d r^d \max\{1, \frac{|E|^d}{r^d}\} = |E|^d,$$

where the last inequality follows from Lemma 5.4. Note that for $0 \leq t \leq s$,

$$\Phi(r)^{s-t} S^t_{\Phi,r}(E) \leq S^s_{\Phi,r}(E) \leq r^{s-t} S^t_{\Phi,r}(E). \tag{5.2}$$

By combining (5.2) with $S^0_{\Phi,r}(E) \geq 1$ and $S^d_{\Phi,r}(E) \lesssim |E|^d$, we can complete the proof in a similar manner like the definition of the Hausdorff dimensions (see [8, Section 3.2]). □

Note that by (5.2) a similar proof of Lemma 2.1 shows that

$$\overline{\dim}_\Phi E = \left(\text{the unique } s \in [0,d] \text{ such that } \limsup_{r\to 0} \frac{\log S^s_{\Phi,r}(E)}{-\log r} = 0\right) \tag{5.3}$$

if $\limsup_{r\to 0} \log r/\log \Phi(r) > 0$, and

(5.4) $$\underline{\dim}_\Phi E = \left(\text{the unique } s \in [0,d] \text{ such that } \liminf_{r\to 0} \frac{\log S^s_{\Phi,r}(E)}{-\log r} = 0\right).$$

if $\liminf_{r\to 0} \log r/\log \Phi(r) > 0$.

5.2. **Generalized capacity dimensions and dimension profiles.** We begin with the introduction of some appropriate kernels in the corresponding settings.

Definition 5.3. (kernels) Let Φ be an admissible function.

- In Setting 1.3, let $0 \le s \le d$ and $0 < r \le 1$. Define

(5.5) $$J^s_{\Phi,r}(x\wedge y) = \max_{\Phi(r)\le u\le r} u^{-s} Z_u(x\wedge y) \quad \text{for } x,y\in\Sigma,$$

  where $Z_u(x\wedge y)$ is defined in (2.1).
- In Setting 1.4 and Setting 1.5, let $\tau > 0$, $0 \le s \le \tau$, and $0 < r \le 1$. Define

(5.6) $$J^{s,\tau}_{\Phi,r}(|x-y|) = \max_{\Phi(r)\le u\le r} u^{-s}\phi^\tau_u(|x-y|) \quad \text{for } x,y\in\mathbb{R}^d,$$

  where

(5.7) $$\phi^\tau_u(\Delta) := \min\left\{1, \left(\frac{u}{\Delta}\right)^\tau\right\} \qquad \text{for } \Delta \ge 0.$$

Like Lemma 3.1, we have the following simple geometric fact.

Lemma 5.4. *Let $B(x,\Delta)\subset\mathbb{R}^m$ be a ball. Then for $r>0$,*

$$N_r(B(x,\Delta)) \lesssim_m \frac{1}{\phi^m_r(\Delta)}.$$

Proof. Write $x=(x_1,\ldots,x_m)$ and $A=\prod_{i=1}^m [x_i-\Delta, x_i+\Delta]$. Note that A can be covered by $C\max\{1,(\Delta/r)^m\}$ manly cubes with side length $r/\sqrt{m}$, where C is a constant only depending on m. This completes the proof since $B(x,\Delta)\subset A$ and the diameter of each cube with side length $r/\sqrt{m}$ is r. □

We proceed by defining the capacities with respect to the above kernels.

Definition 5.5 (capacities)**.** Let X be a compact metric space and $K\colon X\times X\to(0,+\infty)$ be a continuous function. For each compact set $E\subset X$, the *capacity* of E with respect to the *kernel* K is defined by

(5.8) $$C_K(E) := \left(\inf_{\mu\in\mathcal{P}(E)} \iint K(x,y)\, d\mu(x)d\mu(y)\right)^{-1}.$$

By convention we set $C_K(E) = C_K(\overline{E})$ for every non-compact set $E\subset X$. Thus when it comes to capacities, without loss of generality we can assume that the underlying set is compact. In particular, we focus on the following capacities.

- In Setting 1.3, let $K(x,y) = J^s_{\Phi,r}(x\wedge y)$ (see (5.5)). Define

$$C^s_{\Phi,r}(E) := C_K(E) \quad \text{for } E\subset\Sigma.$$

- In Setting 1.4 and Setting 1.5, let $K(x,y)=J^{s,\tau}_{\Phi,r}(|x-y|)$ (see (5.6)). Define
$$C^{s,\tau}_{\Phi,r}(E):=C_K(E)\quad\text{for each bounded set } E\subset\mathbb{R}^d.$$

Now we are ready to define the generalized capacity dimensions called the Φ-*capacity dimensions* and the generalized dimension profiles called the Φ-*dimension profiles*.

Definition 5.6 (Φ-capacity dimensions)**.** In Setting 1.3, let $E\subset\Sigma$. The *upper and lower* Φ-*capacity dimensions* of E are respectively defined by

$$\begin{aligned}\overline{\dim}_{\mathrm{C},\Phi}E&=\inf\{s\geq 0\colon \limsup_{r\to 0} C^s_{\Phi,r}(E)<\infty\}=\inf\{s\geq 0\colon \limsup_{r\to 0} C^s_{\Phi,r}(E)=0\}\\&=\sup\{s\geq 0\colon \limsup_{r\to 0} C^s_{\Phi,r}(E)=\infty\}=\sup\{s\geq 0\colon \limsup_{r\to 0} C^s_{\Phi,r}(E)>0\}\end{aligned}$$

and

$$\begin{aligned}\underline{\dim}_{\mathrm{C},\Phi}E&=\inf\{s\geq 0\colon \liminf_{r\to 0} C^s_{\Phi,r}(E)<\infty\}=\inf\{s\geq 0\colon \liminf_{r\to 0} C^s_{\Phi,r}(E)=0\}\\&=\sup\{s\geq 0\colon \liminf_{r\to 0} C^s_{\Phi,r}(E)=\infty\}=\sup\{s\geq 0\colon \liminf_{r\to 0} C^s_{\Phi,r}(E)>0\}.\end{aligned}$$

Definition 5.7 (Φ-dimension profiles)**.** In Setting 1.4 and Setting 1.5, let $E\subset\mathbb{R}^d$ and $\tau>0$. The *upper and lower* Φ-*dimension profiles* of E are respectively defined by

$$\begin{aligned}\overline{\dim}^{\tau}_{\Phi}E&=\inf\{s\geq 0\colon \limsup_{r\to 0} C^{s,\tau}_{\Phi,r}(E)<\infty\}=\inf\{s\geq 0\colon \limsup_{r\to 0} C^{s,\tau}_{\Phi,r}(E)=0\}\\&=\sup\{s\geq 0\colon \limsup_{r\to 0} C^{s,\tau}_{\Phi,r}(E)=\infty\}=\sup\{s\geq 0\colon \limsup_{r\to 0} C^{s,\tau}_{\Phi,r}(E)>0\}\end{aligned}$$

and

$$\begin{aligned}\underline{\dim}^{\tau}_{\Phi}E&=\inf\{s\geq 0\colon \liminf_{r\to 0} C^{s,\tau}_{\Phi,r}(E)<\infty\}=\inf\{s\geq 0\colon \liminf_{r\to 0} C^{s,\tau}_{\Phi,r}(E)=0\}\\&=\sup\{s\geq 0\colon \liminf_{r\to 0} C^{s,\tau}_{\Phi,r}(E)=\infty\}=\sup\{s\geq 0\colon \liminf_{r\to 0} C^{s,\tau}_{\Phi,r}(E)>0\}.\end{aligned}$$

Definition 5.6 and Definition 5.7 are justified as follows. Let $K^s_{\Phi,r}(x,y)=J^s_{\Phi,r}(x\wedge y)$ or $J^{s,\tau}_{\Phi,r}(|x-y|)$. According to Definition 5.3, for $0\leq t\leq s$,

$$r^{-(s-t)}K^t_{\Phi,r}(x,y)\leq K^s_{\Phi,r}(x,y)\leq \Phi(r)^{-(s-t)}K^t_{\Phi,r}(x,y).$$

Then

$$\Phi(r)^{s-t}C_{K^t_{\Phi,r}}(E)\leq C_{K^s_{\Phi,r}}(E)\leq r^{s-t}C_{K^t_{\Phi,r}}(E).\tag{5.9}$$

Since $J^0_{\Phi,r}(x\wedge y)\leq 1$ and $J^d_{\Phi,r}(x\wedge y)\geq 1$, we have $C^0_{\Phi,r}(E)\geq 1$ and $C^d_{\Phi,r}(E)\leq 1$. Hence

$$0\leq \underline{\dim}_{\mathrm{C},\Phi}E\leq \overline{\dim}_{\mathrm{C},\Phi}E\leq d\quad\text{for } E\subset\Sigma.\tag{5.10}$$

Let $E\subset\mathbb{R}^d$. Since $J^{0,\tau}_{\Phi,r}(|x-y|)\leq 1$ and for $0<r<\min\{1,|E|\}$,

$$J^{\tau,\tau}_{\Phi,r}(|x-y|)=\max_{\Phi(r)\leq u\leq r}\min\left\{\frac{1}{u^\tau},\frac{1}{|x-y|^\tau}\right\}\geq\frac{1}{|E|^\tau}\gtrsim_\tau 1,$$

we have $C^{0,\tau}_{\Phi,r}(E)\geq 1$ and $C^{\tau,\tau}_{\Phi,r}(E)\lesssim 1$ when r is small. Hence

$$0\leq \underline{\dim}^{\tau}_{\Phi}E\leq \overline{\dim}^{\tau}_{\Phi}E\leq\tau.\tag{5.11}$$

A combination of (5.9), (5.10), and (5.11) justifies Definition 5.6 and Definition 5.7 in the same way as the proof of Lemma 5.2.

From (5.9), we can characterize the Φ-capacity dimensions and the Φ-dimension profiles like (5.3) and (5.4) if $\limsup_{r\to 0} \log r/\log \Phi(r) > 0$ and $\liminf_{r\to 0} \log r/\log \Phi(r) > 0$ are respectively assumed.

Before finishing this subsection, we introduce a function Φ_α associated with Φ. It is useful in the proof of Theorem 1.6. For $0 < \alpha \le 1$, define

$$\Phi_\alpha(r) := \Phi(r^\alpha)^{1/\alpha} \quad \text{for } 0 < r < Y^{1/\alpha}. \tag{5.12}$$

It is readily checked that Φ_α is admissible.

5.3. **Upper bound.** We begin with a lemma about the behavior of the capacities under the Hölder continuous maps.

Lemma 5.8. *Let $\tau > 0$ and $0 \le s \le \tau$. Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be a map. If there is some $0 < \alpha \le 1$ such that*

$$|f(x) - f(y)| \lesssim |x - y|^\alpha \quad \textit{for } x, y \in \mathbb{R}^d, \tag{5.13}$$

then for $0 < r \le 1$,

$$J^{s,\tau}_{\Phi,r}(|f(x) - f(y)|) \gtrsim_\tau J^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(|x - y|). \tag{5.14}$$

In particular, for $E \subset \mathbb{R}^d$ and $0 < r \le 1$,

$$C^{s,\tau}_{\Phi,r}(f(E)) \lesssim_\tau C^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(E).$$

Proof. According to (5.6),

$$\begin{aligned} &J^{s,\tau}_{\Phi,r}(|f(x) - f(y)|) \\ &\quad= \max_{\Phi(r)\le u\le r} u^{-s} \min\left\{1, \frac{u^\tau}{|f(x) - f(y)|^\tau}\right\} \\ &\quad\gtrsim_\tau \max_{\Phi(r)\le u\le r} u^{-s} \min\left\{1, \frac{u^\tau}{|x-y|^{\alpha\tau}}\right\} && \text{by (5.13)} \\ &\quad= \max_{\Phi(r)^{1/\alpha}\le v\le r^{1/\alpha}} v^{-\alpha s} \min\left\{1, \frac{v^{\alpha\tau}}{|x-y|^{\alpha\tau}}\right\} && \text{by letting } v = u^{1/\alpha} \\ &\quad= \max_{\Phi_\alpha(r^{1/\alpha})\le v\le r^{1/\alpha}} v^{-\alpha s} \min\left\{1, \frac{v^{\alpha\tau}}{|x-y|^{\alpha\tau}}\right\} && \text{by (5.12)} \\ &\quad= J^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(|x - y|). \end{aligned} \tag{5.15}$$

This proves (5.14).

Without loss of generality we assume that E is compact. By Lemma 2.3, there is an equilibrium measure $\nu \in \mathcal{P}(f(E))$ for the kernel $J^{s,m}_{\Phi,r}(|u - v|)$. Then [23, Theorem 1.20] gives some $\mu \in \mathcal{P}(E)$ such that $\nu = f\mu$. Hence

$$
\begin{aligned}
C^{\alpha s,\alpha\tau}_{\Phi_\alpha,r^{1/\alpha}}(E) &\geq \left(\int J^{\alpha s,\alpha\tau}_{\Phi_\alpha,r^{1/\alpha}}(|x-y|)\,d\mu(x)d\mu(y)\right)^{-1} \\
&\gtrsim_\tau \left(\int J^{s,\tau}_{\Phi,r}(|f(x)-f(y)|)\,d\mu(x)d\mu(y)\right)^{-1} \qquad \text{by (5.14)} \\
&= \left(\int J^{s,\tau}_{\Phi,r}(|u-v|)\,d\nu(u)d\nu(v)\right)^{-1} = C^{s,\tau}_{\Phi,r}(f(E)).
\end{aligned} \tag{5.16}
$$

This completes the proof. □

We now demonstrate how the capacities behave under a map with the modulus of continuity similar to that of the fractional Brownian motion.

Lemma 5.9. *Let $\tau > 0$ and $0 \leq s \leq \tau$. Let $f\colon \mathbb{R}^d \to \mathbb{R}^m$ be a map. If there exist some $0 < \alpha < 1$ and $0 < \Delta < 1$ such that*

$$
|f(x)-f(y)| \lesssim |x-y|^\alpha \log(1/|x-y|) \quad \text{for } x, y \in \mathbb{R}^d \text{ with } |x-y| \leq \Delta, \tag{5.17}
$$

then for all sufficiently small $r > 0$ and $x, y \in \mathbb{R}^d$ with $|x-y| \leq \Delta$,

$$
J^{s,\tau}_{\Phi,r}(|f(x)-f(y)|) \gtrsim_{\tau,\alpha} [\log(1/\Phi(r))]^{-\tau} J^{\alpha s,\alpha\tau}_{\Phi_\alpha,r^{1/\alpha}}(|x-y|). \tag{5.18}
$$

Let $E \subset \mathbb{R}^d$ be a bounded set. Suppose further that there is some $0 < \beta \leq 1$ such that

$$
|f(x)-f(y)| \lesssim |x-y|^\beta \quad \text{for } x, y \in E. \tag{5.19}
$$

Then for all sufficiently small $r > 0$,

$$
C^{s,\tau}_{\Phi,r}(f(E)) \lesssim_{\tau,\alpha,\beta} [\log(1/\Phi(r))]^\tau C^{\alpha s,\alpha m}_{\Phi_\alpha,r^{1/\alpha}}(E). \tag{5.20}
$$

Proof. Let $x, h \in \mathbb{R}^d$ with $|h| \leq \Delta$. According to (5.6),

$$
\begin{aligned}
&J^{s,\tau}_{\Phi,r}(|f(x+h)-f(x)|) \\
&\quad= \max_{\Phi(r)\leq u\leq r} u^{-s}\min\left\{1, \frac{u^\tau}{|f(x+h)-f(x)|^\tau}\right\} \\
&\quad\gtrsim_\tau \max_{\Phi(r)\leq u\leq r} u^{-s}\min\left\{1, \frac{u^\tau}{|h|^{\alpha\tau}[\log(1/|h|)]^\tau}\right\} \qquad \text{by (5.17)} \\
&\quad= \max_{\Phi(r)\leq u\leq r} u^{-s}\min\left\{1, \left(\frac{u}{-|h|^\alpha \log|h|}\right)^\tau\right\}.
\end{aligned} \tag{5.21}
$$

For $\Phi(r) \leq u \leq r$, write

$$
L(u,h) := \min\left\{1, \left(\frac{u}{-|h|^\alpha\log|h|}\right)^\tau\right\} \quad \text{and} \quad M(u,h) := \min\left\{1, \frac{u^\tau}{|h|^{\alpha\tau}}\right\}.
$$

If $u \geq -|h|^\alpha \log|h|$, then $L(u,h) = 1 \geq M(u,h)$; If $u < -|h|^\alpha\log|h|$, then $\Phi(r) \leq -|h|^\alpha \log|h|$. Denote $g(x) := -x^\alpha \log x$ for $x > 0$. Note that when r is small,

$$
g\left(\Phi(r)^{2/\alpha}\right) = -\frac{2}{\alpha}\Phi(r)^2 \log\Phi(r) \leq \Phi(r) \leq g(|h|).
$$

Since $g(x)$ is increasing on $(0, \exp(-1/\alpha))$, we have $|h| \geq \Phi(r)^{2/\alpha}$. Hence

$$L(u,h) = \left(\frac{u}{-|h|^\alpha \log|h|}\right)^\tau \geq \left(\frac{\alpha}{2}\right)^\tau [\log(1/\Phi(r))]^{-\tau} \frac{u^\tau}{|h|^{\alpha\tau}} \gtrsim_{\alpha,\tau} [\log(1/\Phi(r))]^{-\tau} M(u,h).$$

This concludes that

$$L(u,h) \gtrsim_{\alpha,\tau} [\log(1/\Phi(r))]^{-\tau} M(u,h) \quad \text{for } \Phi(r) \leq u \leq r,\ |h| \leq \Delta.$$

Together with (5.21), we have

$$\begin{aligned} J^{s,\tau}_{\Phi,r}(|f(x+h)-f(x)|) &\gtrsim_{\tau,\alpha} [\log(1/\Phi(r))]^{-\tau} \max_{\Phi(r)\leq u\leq r} u^{-s} \min\left\{1, \frac{u^\tau}{|h|^{\alpha\tau}}\right\} \\ &= [\log(1/\Phi(r))]^{-\tau} J^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(|h|), \end{aligned}$$

where the last equality is by taking $v = u^{1/\alpha}$. This proves (5.18).

Without loss of generality assume that E is compact. For $x, y \in E$ with $|x-y| \geq \Delta$, (5.19) implies that

$$|f(x)-f(y)| \lesssim |E|^\beta = \frac{|E|^\beta}{\Delta^\alpha}\Delta^\alpha \lesssim_{\alpha,\beta} \Delta^\alpha \leq |x-y|^\alpha,$$

thus the calculation in (5.15) shows that for $x, y \in E$ with $|x-y| \geq \Delta$,

$$J^{s,\tau}_{\Phi,r}(|f(x)-f(y)|) \gtrsim_{\tau,\alpha,\beta} J^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(|x-y|).$$

Together with (5.18), we have for $x, y \in E$ and sufficiently small $r > 0$,

$$J^{s,\tau}_{\Phi,r}(|f(x)-f(y)|) \gtrsim_{\tau,\alpha,\beta} [\log(1/\Phi(r))]^{-\tau} J^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(|x-y|). \tag{5.22}$$

Based on (5.22), we apply the arguments in (5.16) to get

$$C^{s,\tau}_{\Phi,r}(f(E)) \lesssim_{\tau,\alpha,\beta} [\log(1/\Phi(r))]^{\tau} C^{\alpha s,\alpha\tau}_{\Phi_\alpha, r^{1/\alpha}}(E).$$

This finishes the proof. □

To apply Lemma 5.9, below we recall Lévy's modulus of continuity for the fractional Brownian motion (see e.g., [22, Chp. 18, Eq. (3)]).

Lemma 5.10. *In Setting 1.5, let $E \subset \mathbb{R}^d$ be a bounded set. Then almost surely,*

$$|B_\alpha(x) - B_\alpha(y)| \lesssim |x-y|^{\alpha/2} \quad \text{for } x, y \in E,$$

and there exists some $0 < \Delta < 1/10$ such that

$$|B_\alpha(x) - B_\alpha(y)| \lesssim |x-y|^\alpha \sqrt{\log(1/|x-y|)} \quad \text{for } x, y \in E \text{ with } |x-y| \leq \Delta.$$

Remark 5.11. In [5, 9], the Hölder continuity of the fractional Brownian motion is sufficient for obtaining results about the θ-intermediate dimensions. However, for the Φ-intermediate dimensions, the more precise modulus of continuity in Lemma 5.10 seems necessary.

Now we are ready to prove the key ingredients in the proof of the upper-bound part of Theorem 1.6. They are analogous to Proposition 3.2.

Proposition 5.12. *Let Φ be an admissible function.*

(i) In Setting 1.3, let $E \subset \Sigma$ be compact and $0 \le s \le d$. Let $\mathbf{a} \in \mathbb{R}^{dm}$. If for $0 < r \le 1$ there exist $\mu \in \mathcal{P}(E)$ and $\gamma > 0$ such that

$$\int J^s_{\Phi,r}(x \wedge y)\, d\mu(y) \ge \gamma \quad \text{for all } x \in E$$

then for all sufficiently small $r > 0$,

$$S^s_{\Phi,r}(\pi^{\mathbf{a}}(E)) \lesssim \frac{\log(1/\Phi(r))}{\gamma}.$$

In particular,

$$S^s_{\Phi,r}(\pi^{\mathbf{a}}(E)) \lesssim_{d,\mathbf{a}} \log(1/\Phi(r)) C^s_{\Phi,r}(E). \tag{5.23}$$

(ii) In Setting 1.4, let $F \subset \mathbb{R}^m$ be compact and $0 \le s \le m$. If for $0 < r \le 1$ there exist $\mu \in \mathcal{P}(F)$ and $\gamma > 0$ such that

$$\int J^{s,m}_{\Phi,r}(|x-y|)\, d\mu(y) \ge \gamma \quad \text{for all } x \in F, \tag{5.24}$$

then for all sufficiently small $r > 0$,

$$S^s_{\Phi,r}(F) \lesssim \frac{\log(1/\Phi(r))}{\gamma}. \tag{5.25}$$

In particular, for $E \subset \mathbb{R}^d$ and $V \in G(d,m)$,

$$S^s_{\Phi,r}(P_V E) \lesssim \log(1/\Phi(r))\, C^{s,m}_{\Phi,r}(E). \tag{5.26}$$

(iii) In Setting 1.5, let $E \subset \mathbb{R}^d$ be compact and $0 \le s \le m$. Then almost surely for all sufficiently small $r > 0$,

$$S^s_{\Phi,r}(B_\alpha(E)) \lesssim [\log(1/\Phi(r))]^{m+1} C^{\alpha s, \alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E). \tag{5.27}$$

Proof. (i) follows from a similar proof of Proposition 3.2.

Next we prove (ii) by adapting some ideas of [6] but considering a different kernel $J^{s,m}_{\Phi,r}(|x-y|)$. Write $D := \lceil \log(|F|/\Phi(r)) \rceil$. Let $x \in F$. Then $F \subset B(x, |F|) \subset B(x, \exp(D)\,\Phi(r))$. For simplicity, define

$$\delta_k = \exp(k)\Phi(r) \quad \text{for } k = 0, 1, 2, \ldots$$

By convention we set $\delta_{-1} := \Phi(r)$.

Since $J^{s,m}_{\Phi,r}(|x-y|) = \Phi(r)^{-s}$ for $y \in B(x, \Phi(r))$ and $\Delta \mapsto J^{s,m}_{\Phi,r}(\Delta)$ is non-increasing, by (5.24),

$$\begin{aligned}
\gamma &\le \int J^{s,m}_{\Phi,r}(|x-y|)\, d\mu(y) \\
&= \int_{B(x,\delta_0)} J^{s,m}_{\Phi,r}(|x-y|)\, d\mu(y) + \sum_{k=1}^{D} \int_{B(x,\delta_k)\setminus B(x,\delta_{k-1})} J^{s,m}_{\Phi,r}(|x-y|)\, d\mu(y) \\
&\le \sum_{k=0}^{D} J^{s,m}_{\Phi,r}(\delta_{k-1}) \mu(B(x,\delta_k)).
\end{aligned}$$

Hence for each $x \in F$, there exists some integer $k(x) \in [0, D]$ such that

$$J^{s,m}_{\Phi,r}(\delta_{k(x)-1})\mu(B(x,\delta_{k(x)})) \geq \frac{\gamma}{D+1}. \tag{5.28}$$

By (5.6), we can find some $u(x) \in [\Phi(r), r]$ such that

$$u(x)^{-s}\phi^m_{u(x)}(\delta_{k(x)-1}) = J^{s,m}_{\Phi,r}(\delta_{k(x)-1}). \tag{5.29}$$

Since

$$\phi^m_{u(x)}(\delta_{k(x)}) \geq \exp(-m)\phi^m_{u(x)}(\delta_{k(x)-1}) \gtrsim_m \phi^m_{u(x)}(\delta_{k(x)-1}),$$

it follows from (5.28) and (5.29) that

$$u(x)^{-s}\phi^m_{u(x)}(\delta_{k(x)})\mu(B(x,\delta_{k(x)})) \gtrsim_m \frac{\gamma}{D+1}.$$

By a rearrangement,

$$\frac{u(x)^s}{\phi^m_{u(x)}(\delta_{k(x)})} \lesssim_m \frac{D+1}{\gamma}\mu(B(x,\delta_{k(x)})). \tag{5.30}$$

Let $\mathscr{B} := \{B(x,\delta_{k(x)})\}_{x\in F}$. Then $\mathscr{B}$ is a cover of F. For each $B = B(x,\delta_{k(x)}) \in \mathscr{B}$, write $\delta_B := \delta_{k(x)}$ and $u_B := u(x) \in [\Phi(r), r]$. Then (5.30) becomes

$$\frac{u_B^s}{\phi^m_{u_B}(\delta_B)} \lesssim_m \frac{D+1}{\gamma}\mu(B) \quad \text{for } B \in \mathscr{B}. \tag{5.31}$$

By Lemma 5.4, for each $B \in \mathscr{B}$ there is a cover Γ_B consisting of sets with diameter u_B such that

$$\#\Gamma_B \lesssim_m \frac{1}{\phi^m_{u_B}(\delta_B)}. \tag{5.32}$$

By Besicovitch covering theorem (see [23, Theorem 2.7]), there are $\mathscr{B}_1, \dots, \mathscr{B}_c \subset \mathscr{B}$ covering F such that each $\mathscr{B}_i$ is disjoint, where c only depends on m, that is,

$$F \subset \bigcup_{i=1}^{c} \bigcup_{B\in\mathscr{B}_i} B$$

and

$$B \cap B' = \emptyset \quad \text{for } B, B' \in \mathscr{B}_i,\ B \neq B',\ i = 1, \dots, c.$$

Then the collection $\bigcup_{i=1}^c \bigcup_{B\in\mathscr{B}_i} \Gamma_B$ covers F and consists of sets with diameters in $[\Phi(r), r]$. Hence

$$\begin{aligned}
S^s_{\Phi,r}(F) &\leq \sum_{i=1}^{c} \sum_{B\in\mathscr{B}_i} \sum_{U\in\Gamma_B} |U|^s \\
&= \sum_{i=1}^{c} \sum_{B\in\mathscr{B}_i} \#\Gamma_B \cdot u_B^s \\
&\lesssim_m \sum_{i=1}^{c} \sum_{B\in\mathscr{B}_i} \frac{u_B^s}{\phi^m_{u_B}(\delta_B)} && \text{by (5.32)} \\
&\lesssim_m \sum_{i=1}^{c} \sum_{B\in\mathscr{B}_i} \frac{D+1}{\gamma}\mu(B) && \text{by (5.31)}
\end{aligned}$$

$$\lesssim_m \frac{D+1}{\gamma} \qquad \text{by each } \mathscr{B}_i \text{ disjoint}$$
$$\lesssim \frac{\log(1/\Phi(r))}{\gamma}$$

when r is small. This proves (5.25).

Since Lemma 2.3 gives an equilibrium measure $\mu \in \mathcal{P}(F)$ satisfying (5.24) with $\gamma = 1/C^{s,m}_{\Phi,r}(F)$, it follows from (5.25) that

$$S^s_{\Phi,r}(F) \lesssim \log(1/\Phi(r)) C^{s,m}_{\Phi,r}(F). \tag{5.33}$$

Since $|P_V x - P_V y| \leq |x-y|$ for each orthogonal projection P_V, applying Lemma 5.8 with $\alpha = 1$ shows

$$C^{s,m}_{\Phi,r}(P_V E) \lesssim C^{s,m}_{\Phi,r}(E). \tag{5.34}$$

Applying (5.33) with $F = P_V E$, we obtain (5.26) from (5.34).

Finally we move to (iii). By Lemma 5.10 and Lemma 5.9, we have almost surely that

$$C^{s,m}_{\Phi,r}(B_\alpha(E)) \lesssim_{m,\alpha} [\log(1/\Phi(r))]^m C^{\alpha s, \alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E).$$

Hence applying (5.33) with $F = B_\alpha(E)$ gives that almost surely,

$$S^s_{\Phi,r}(B_\alpha(E)) \lesssim [\log(1/\Phi(r))]^{m+1} C^{\alpha s, \alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E).$$

This completes the proof. □

5.4. **Lower bound.** We begin with an analog of Lemma 4.1. For $0 \leq s \leq d$ and $r > 0$, define

$$\psi^s_{\Phi,r}(\Delta) = \begin{cases} \Phi(r)^{-s} & \Delta \leq \Phi(r) \\ \Delta^{-s} & \Phi(r) < \Delta \leq r \\ 0 & \Delta > r \end{cases} \quad \text{for } \Delta \geq 0. \tag{5.35}$$

Lemma 5.13. *Let Φ be an admissible function. Let $0 \leq s \leq d$, $0 < r \leq 1$, and $E \subset \mathbb{R}^d$ be a non-empty compact set. If there exist $\mu \in \mathcal{P}(E)$, a Borel subset $F \subset E$, and $\gamma > 0$ such that*

$$\int \psi^s_{\Phi,r}(|x-y|)\, d\mu(y) \leq \gamma \quad \text{for all } x \in F,$$

then

$$S^s_{\Phi,r}(E) \geq \frac{\mu(F)}{\gamma}.$$

Lemma 5.13 follows from a similar proof of Lemma 4.1. Below we provide two lemmas showing that there are some appropriate transversalities in Setting 1.4 and Setting 1.5, which are the analogs of Lemma 4.2.

Lemma 5.14 ([23, Lemma 3.11])**.** *In Setting 1.4, let $x, y \in \mathbb{R}^d$ and $r > 0$. Then*

$$\gamma_{d,m}\{V \in G(d,m) \colon |P_V x - P_V y| \leq r\} \lesssim_{d,m} \phi^m_r(|x-y|) \tag{5.36}$$

where $\phi^m_r(|x-y|)$ is as in (5.7).

Lemma 5.15. *In Setting 1.5, let $x, y \in \mathbb{R}^d$ and $r > 0$. Then*

$$\mathbb{P}\{|B_\alpha(x) - B_\alpha(y)| \le r\} \lesssim_m \phi^{\alpha m}_{r^{1/\alpha}}(|x-y|). \tag{5.37}$$

Proof. By (1.2),

$$\begin{aligned}
\mathbb{P}\{|B_\alpha(x) - B_\alpha(y)| \le r\} &\le \mathbb{P}\{|B_{\alpha,i}(x) - B_{\alpha,i}(y)| \le r \text{ for all } 1 \le i \le m\} \\
&= \left(\frac{1}{\sqrt{2\pi}|x-y|^\alpha} \int_{|t|\le r} \exp\left(-\frac{t^2}{2|x-y|^{2\alpha}}\right) dt \right)^m \\
&\le \left(\frac{1}{|x-y|^\alpha} \int_{|t|\le r} 1\, dt \right)^m \\
&= 2^m \frac{r^m}{|x-y|^{\alpha m}} \\
&\lesssim_m \left(\frac{r^{1/\alpha}}{|x-y|} \right)^{\alpha m}.
\end{aligned}$$

Since $\mathbb{P}(A) \le 1$ for all events $A \subset \Omega$, we have

$$\mathbb{P}\{|B_\alpha(x) - B_\alpha(y)| \le r\} \lesssim_m \min\left\{1, \left(\frac{r^{1/\alpha}}{|x-y|}\right)^{\alpha m}\right\} = \phi^{\alpha m}_{r^{1/\alpha}}(|x-y|).$$

This finishes the proof. □

As an analog of Proposition 4.3, the following lemma reveals a unified computational scheme for the integrals over parameters in various contexts.

Proposition 5.16. *Let Φ be an admissible function and $0 < r \le 1$.*

(i) In Setting 1.3, assume $\|T_j\| < 1/2$ for $1 \le j \le m$. Let $0 \le s \le d$ and $x, y \in \Sigma$. Then

$$\int_{B_\rho} \psi^s_{\Phi,r}(|\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)|)\, d\mathbf{a} \lesssim_{\rho,d} \log(1/\Phi(r)) J^s_{\Phi,r}(x \wedge y)$$

where B_ρ denotes the closed ball in $\mathbb{R}^{dm}$ centered at 0 with radius $\rho > 0$.

(ii) In Setting 1.4, let $0 \le s \le m$ and $x, y \in \mathbb{R}^d$. Then

$$\int_{G(d,m)} \psi^s_{\Phi,r}(|P_V x - P_V y|)\, d\gamma_{d,m}(V) \lesssim_{d,m} \log(1/\Phi(r)) J^{s,m}_{\Phi,r}(|x-y|).$$

(iii) In Setting 1.5, let $0 \le s \le m$ and $x, y \in \mathbb{R}^d$. Then

$$\int_\Omega \psi^s_{\Phi,r}(|B_\alpha(x) - B_\alpha(y)|)\, d\mathbb{P}(\omega) \lesssim_m \log(1/\Phi(r)) J^{\alpha s,\alpha m}_{\Phi_\alpha, r^{1/\alpha}}(|x-y|).$$

Proof. We begin with a general computational scheme assuming the abstract transversality (5.38). Then the proof is completed by substituting (5.38) with the corresponding transversality in different settings.

Let (Λ, ν) be a measure space and $\lambda \mapsto \Delta_\lambda$ be a measurable function from Λ to $(0, +\infty)$. Suppose

$$\nu\{\lambda \colon \Delta_\lambda \le r\} \lesssim K_r \quad \text{for } r > 0, \tag{5.38}$$

where $r \mapsto K_r$ is a measurable function. According to (5.35),

$$\begin{aligned}
&\int_\Lambda \psi^s_{\Phi,r}(\Delta_\lambda)\, d\nu(\lambda)\\
&\quad = \int_{\{\lambda\colon \Delta_\lambda \le \Phi(r)\}} \Phi(r)^{-s}\, d\nu(\lambda) + \int_{\{\lambda\colon \Phi(r)<\Delta_\lambda\le r\}} \Delta_\lambda^{-s}\, d\nu(\lambda)\\
&\quad = \Phi(r)^{-s}\nu\{\lambda\colon \Delta_\lambda \le \Phi(r)\} + \int_0^\infty \nu\{\lambda\colon \Phi(r) < \Delta_\lambda \le r,\ \Delta_\lambda^{-s} \ge t\}\, dt\\
&\quad = \Phi(r)^{-s}\nu\{\lambda\colon \Delta_\lambda \le \Phi(r)\} + \int_0^{\Phi(r)^{-s}} \nu\{\lambda\colon \Phi(r) < \Delta_\lambda \le \min\{r, t^{-1/s}\}\}\, dt\\
&\quad = \int_0^{\Phi(r)^{-s}} \nu\{\lambda\colon \Delta_\lambda \le \min\{r, t^{-1/s}\}\}\, dt\\
&\quad = \int_0^{r^{-s}} \nu\{\lambda\colon \Delta_\lambda \le r\}\, dt + \int_{r^{-s}}^{\Phi(r)^{-s}} \nu\{\lambda\colon \Delta_\lambda \le t^{-1/s}\}\, dt\\
&\quad = r^{-s}\nu\{\lambda\colon \Delta_\lambda \le r\} + \int_{r^{-s}}^{\Phi(r)^{-s}} \nu\{\lambda\colon \Delta_\lambda \le t^{-1/s}\}\, dt.
\end{aligned}$$

By changing variable with $u = t^{-1/s}$,

$$\begin{aligned}
&\int_\Lambda \psi^s_{\Phi,r}(\Delta_\lambda)\, d\nu(\lambda)\\
&\quad = r^{-s}\nu\{\lambda\colon \Delta_\lambda \le r\} + s\int_{\Phi(r)}^{r} u^{-(s+1)}\nu\{\lambda\colon \Delta_\lambda \le u\}\, du\\
&\quad \lesssim r^{-s}K_r + s\int_{\Phi(r)}^{r} u^{-(s+1)}K_u\, du \qquad \text{by (5.38)} \qquad (5.39)\\
&\quad = \left(1 + s\int_{\Phi(r)}^{r} u^{-1}\, du\right) \max_{\Phi(r)\le u\le r} u^{-s}K_u\\
&\quad \le (s+1)\log(1/\Phi(r)) \left(\max_{\Phi(r)\le u\le r} u^{-s}K_u\right),
\end{aligned}$$

where the last inequality follows from $\int_{\Phi(r)}^r u^{-1}\, du \le \log(1/\Phi(r))$.

Finally by replacing (5.38) with Lemma 4.2, Lemma 5.14, and Lemma 5.15 respectively, we finish the proof by (5.39). □

5.5. **Proof of Theorem 1.6.**

Proof of Theorem 1.6. Based on Proposition 5.12 and Proposition 5.16, the statements of different settings in Theorem 1.6 result from similar arguments. Hence to avoid repetitions while maintaining clarity, we exemplify the arguments by showing (1.4) and (1.5). Without loss of generality, we assume that E is compact.

For (1.4), we show $\underline{\dim}_\Phi P_V E \le \underline{\dim}^m_\Phi E$ while the proof of $\overline{\dim}_\Phi P_V E \le \overline{\dim}^m_\Phi E$ is similar. Let $s > t > \underline{\dim}^m_\Phi E$. Then

$$\liminf_{r\to 0} C^{t,m}_{\Phi,r}(E) = 0. \qquad (5.40)$$

By (1.3), there is some $A > 0$ such that

$$r^{s-t}\log(1/\Phi(r)) \le A \quad \text{for } r > 0.$$

Hence by (ii) of Proposition 5.12 and (5.9),

$$S^s_{\Phi,r}(P_V E) \lesssim \log(1/\Phi(r))C^{s,m}_{\Phi,r}(E) \le r^{s-t}\log(1/\Phi(r))C^{t,m}_{\Phi,r}(E) \le A\, C^{t,m}_{\Phi,r}(E).$$

By (5.40), taking $\liminf_{r\to 0}$ on both sides of the above inequality implies

$$\liminf_{r\to 0} S^s_{\Phi,r}(P_V E) = 0.$$

This shows $\underline{\dim}_\Phi P_V E \le s$ by Lemma 5.2. Letting $s \to \underline{\dim}^m_\Phi E$ gives

$$\underline{\dim}_\Phi P_V E \le \underline{\dim}^m_\Phi E.$$

Next we prove (1.5) by showing that almost surely $\overline{\dim}_\Phi B_\alpha(E) = \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E$ while the proof for almost surely $\underline{\dim}_\Phi B_\alpha(E) = \frac{1}{\alpha}\underline{\dim}^{\alpha m}_{\Phi_\alpha} E$ is similar. Let $s > t > \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E$. Then

$$\limsup_{r\to 0} C^{\alpha t,\alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E) = \limsup_{r\to 0} C^{\alpha t,\alpha m}_{\Phi_\alpha, r}(E) = 0. \tag{5.41}$$

By (1.3), there is some $A > 0$ such that

$$r^{(s-t)/(m+1)}\log(1/\Phi(r)) \le A \quad \text{for } r > 0. \tag{5.42}$$

By (iii) of Proposition 5.12, almost surely,

$$\begin{aligned}
S^s_{\Phi,r}(B_\alpha(E)) &\lesssim [\log(1/\Phi(r))]^{m+1} C^{\alpha s,\alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E) \\
&\le r^{(s-t)}[\log(1/\Phi(r))]^{m+1} C^{\alpha t,\alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E) && \text{by (5.9)} \\
&\le A^{m+1}\, C^{\alpha t,\alpha m}_{\Phi_\alpha, r^{1/\alpha}}(E) && \text{by (5.42).}
\end{aligned}$$

By taking $\limsup_{r\to 0}$ on both sides, it follows from (5.41) that almost surely,

$$\limsup_{r\to 0} S^s_{\Phi,r}(B_\alpha(E)) = 0.$$

Then $\overline{\dim}_\Phi B_\alpha(E) \le s$ by Lemma 5.2. Letting $s \to \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E$ gives

$$\overline{\dim}_\Phi B_\alpha(E) \le \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E.$$

Hence it suffices to prove that almost surely

$$\overline{\dim}_\Phi B_\alpha(E) \ge \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E. \tag{5.43}$$

Suppose $\frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E > 0$, otherwise (5.43) holds trivially. Let $t < s < \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E$. Take a sequence (r_k) tending to 0 such that $0 < r_k \le 2^{-k}$ and

$$\limsup_{k\to\infty} C^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(E) = \limsup_{r\to 0} C^{\alpha s,\alpha m}_{\Phi_\alpha, r}(E) > 0. \tag{5.44}$$

By Lemma 2.3, for each $k \in \mathbb{N}$ there is an equilibrium measure μ_k on E for the kernel $J^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(|x-y|)$. Write

$$\gamma_k := \frac{1}{C^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(E)} = \iint J^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(|x-y|)\, d\mu_k(x)d\mu_k(y).$$

By (iii) of Proposition 5.16,

$$\iint \int_\Omega \psi^s_{\Phi,r_k}(|B_\alpha(x)-B_\alpha(y)|)\,d\mathbb{P}(\omega)\,d\mu_k(x)d\mu_k(y) \lesssim \log(1/\Phi(r_k))\gamma_k. \tag{5.45}$$

Set $\varepsilon := s-t$. By (1.3), there is some $A>0$ such that

$$r^{\varepsilon/2}\log(1/\Phi(r)) \le A \quad \text{for } r>0. \tag{5.46}$$

Then summing (5.45) over $k\in\mathbb{N}$ and using Fubini's theorem lead to

$$\begin{aligned}
&\int_\Omega \sum_{k=1}^\infty \left(r_k^\varepsilon \gamma_k^{-1} \iint \psi^s_{\Phi,r_k}(|B_\alpha(x)-B_\alpha(y)|)\,d\mu_k(x)d\mu_k(y)\right) d\mathbb{P}(\omega)\\
&\quad\lesssim \sum_{k=1}^\infty \log(1/\Phi(r_k))r_k^\varepsilon && \text{by (5.45)}\\
&\quad\le A\sum_{k=1}^\infty r_k^{\varepsilon/2} && \text{by (5.46)}\\
&\quad\le A\sum_{k=1}^\infty 2^{-k\varepsilon/2} < \infty && \text{by } r_k\le 2^{-k}.
\end{aligned}$$

Hence almost surely there exists $M>0$ such that

$$\iint \psi^s_{\Phi,r_k}(|u-v|)\,dB_\alpha\mu_k(v)dB_\alpha\mu_k(u) \le M\gamma_k r_k^{-\varepsilon} \quad \text{for all } k\in\mathbb{N}.$$

Then for each $k\in\mathbb{N}$ there is some $F_k\subset B_\alpha(E)$ such that $(B_\alpha\mu_k)(F_k)\ge 1/2$ and

$$\int \psi^s_{\Phi,r_k}(|u-v|)\,dB_\alpha\mu_k(v) \le 2M\gamma_k r_k^{-\varepsilon} \quad \text{for all } u\in F_k.$$

It follows from Lemma 5.13 that for each $k\in\mathbb{N}$,

$$S^s_{\Phi,r_k}(B_\alpha(E)) \ge \frac{1}{2}(2M\gamma_k r_k^{-\varepsilon})^{-1} \gtrsim r_k^\varepsilon\gamma_k^{-1} = r_k^\varepsilon\, C^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(E). \tag{5.47}$$

Finally, almost surely we have

$$\begin{aligned}
\limsup_{r\to 0} S^t_{\Phi,r}(B_\alpha(E)) &\ge \limsup_{k\to\infty} S^t_{\Phi,r_k}(B_\alpha(E))\\
&\ge \limsup_{k\to\infty} r_k^{-(s-t)} S^s_{\Phi,r_k}(B_\alpha(E)) && \text{by (5.2)}\\
&\gtrsim \limsup_{k\to\infty} r_k^{-(s-t)} r_k^\varepsilon\, C^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(E) && \text{by (5.47)}\\
&= \limsup_{k\to\infty} C^{\alpha s,\alpha m}_{\Phi_\alpha, r_k^{1/\alpha}}(E) && \text{by } \varepsilon = s-t\\
&> 0 && \text{by (5.44)}
\end{aligned}$$

This shows that almost surely,

$$\overline{\dim}_\Phi B_\alpha(E) \ge t.$$

Letting $t\to \frac{1}{\alpha}\overline{\dim}^{\alpha m}_{\Phi_\alpha} E$ gives (5.43). □

6. Final remarks

In the section we give a few remarks.

Firstly, we give a specific example such that the equalities in Theorem 1.2(ii) hold.

Example 6.1. Let $\mathcal{F} = \{f_j(x) = T_j x + a_j\}_{j=1}^m$ be a IFS consisting of similarities, that is, $T_j = \lambda_j O_j$ for some $0 < \lambda_j < 1$ and orthogonal matrix O_j. Let K and π be respectively the self-affine set and the coding map for $\mathcal{F}$. Suppose that $\mathcal{F}$ satisfies the strong separation condition (SSC), that is, $f_i(K) \cap f_j(K) = \emptyset$ for $i \neq j$. Then for $0 < \theta \leq 1$ and $E \subset \Sigma$,

$$\underline{\dim}_\theta\, \pi(E) = \underline{\dim}_{C,\theta}\, E \quad \text{and} \quad \overline{\dim}_\theta\, \pi(E) = \overline{\dim}_{C,\theta}\, E. \tag{6.1}$$

Next we briefly justify Example 6.1 for $\overline{\dim}_\theta$, and it is similar for $\underline{\dim}_\theta$. Take $\Phi(r) = r^{1/\theta}$ and write $J^{s,d}_{\theta,r}(\cdot) = J^{s,d}_{\Phi,r}(\cdot)$, $C^{s,d}_{\theta,r}(\cdot) = C^{s,d}_{\Phi,r}(\cdot)$, and $\overline{\dim}^d_\theta = \overline{\dim}^d_\Phi$. Then

$$C^{s,d}_{\theta,r}(F) \leq S^s_{\theta,r}(F) \lesssim_\theta \log(1/r) C^{s,d}_{\theta,r}(F) \quad \text{for } F \subset \mathbb{R}^d, \tag{6.2}$$

where the first inequality is by Lemma 4.1 and $\psi^s_{\theta,r}(\cdot) \leq J^{s,d}_{\theta,r}(\cdot)$ while the second inequality is by (5.26) with $\Phi(r) = r^{1/\theta}$ and $V = \mathbb{R}^d$. By Lemma 2.1 and Definition 5.7, it follows from (6.2) that

$$\overline{\dim}_\theta F = \overline{\dim}^d_\theta\, F \quad \text{for } F \subset \mathbb{R}^d. \tag{6.3}$$

Let $x, y \in \Sigma$. Since $\mathcal{F}$ consists of similarities and satisfies SSC,

$$|\pi(x) - \pi(y)| \approx \|T_{x\wedge y}\|.$$

Then

$$\begin{aligned} Z_r(x \wedge y) &= \min\left\{1, \left(\frac{r}{\|T_{x\wedge y}\|}\right)^d\right\} \\ &\approx \min\left\{1, \left(\frac{r}{|\pi(x) - \pi(y)|}\right)^d\right\} = \phi^d_r(|\pi(x) - \pi(y)|), \end{aligned}$$

and so by Definition 5.3,

$$J^{s,d}_{\theta,r}(|\pi(x) - \pi(y)|) \approx J^s_{\theta,r}(x \wedge y).$$

Since π is bi-Lipschitz with respect to the metric $d(x,y) = \|T_{x\wedge y}\|$ for $x, y \in \Sigma$,

$$C^{s,d}_{\theta,r}(\pi(E)) \approx C^s_{\theta,r}(E) \quad \text{for } E \subset \Sigma.$$

Hence by Definition 5.6 and Definition 5.7,

$$\overline{\dim}^d_\theta\, \pi(E) = \overline{\dim}_{C,\theta}\, E \quad \text{for } E \subset \Sigma. \tag{6.4}$$

Combining (6.3) and (6.4) gives (6.1).

In [1, Definition 2.7], the admissibility of Φ is assumed in the definitions of the Φ-intermediate dimensions in some general metric spaces. However, in Theorem 1.6 concerning the Φ-intermediate dimensions in $\mathbb{R}^d$, we may only require that Φ is monotone and satisfies $0 < \Phi(r) \leq r$ instead of the admissibility.

There is no obstruction in adapting the arguments in [14, Section 9] to estimate the Hausdorff dimensions of the exceptional sets for the Φ-intermediate dimensions. For example, below we give one such result.

Proposition 6.2. *In Setting 1.3, assume $\|T_j\| < 1/2$ for $1 \le j \le m$. Let Φ be an admissible function satisfying (1.3). Then for $E \subset \Sigma$ and $0 < \delta < d$,*

$$\dim_{\mathrm{H}}\{\mathbf{a} \in \mathbb{R}^{dm} \colon \underline{\dim}_{\Phi}\pi^{\mathbf{a}}(E) < \underline{\dim}_{\mathrm{C},\Phi}E - \delta\} \le dm - \delta,$$

and

$$\dim_{\mathrm{H}}\{\mathbf{a} \in \mathbb{R}^{dm} \colon \overline{\dim}_{\Phi}\pi^{\mathbf{a}}(E) < \overline{\dim}_{\mathrm{C},\Phi}E - \delta\} \le dm - \delta.$$

Inspired by [6, Corollary 6.4] and [1, Theorem 6.1], we can deduce an interesting corollary from Theorem 1.6 by proving the analogs of the corollaries in [6, Section 6].

Corollary 6.3. *Let $E \subset \mathbb{R}^d$ be a bounded set. Suppose there is a family of admissible functions $\{\Psi_s\}$ such that $\underline{\dim}_{\Psi_s}E = s$ and Ψ_s satisfies (1.3) for $s \in [\dim_{\mathrm{H}} E, \underline{\dim}_{\mathrm{B}}E]$. Then $\underline{\dim}_{\mathrm{B}}P_V E = m$ for $\gamma_{d,m}$-a.e. $V \in G(d,m)$ if and only if $\dim_{\mathrm{H}} E \ge m$.*

A similar result holds for the upper dimensions replacing $\underline{\dim}_{\Psi}E$ and $\underline{\dim}_{\mathrm{B}}E$ with $\overline{\dim}_{\Psi}E$ and $\overline{\dim}_{\mathrm{B}}E$, respectively.

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