

ON THE COINCIDENCE OF THE HAUSDORFF AND BOX DIMENSIONS FOR SOME AFFINE-INVARIANT SETS

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ABSTRACT. Let K be a compact subset of the d -torus invariant under an expanding diagonal endomorphism with s distinct eigenvalues. Suppose the symbolic coding of K satisfies weak specification. When $s \leq 2$, we prove that the following three statements are equivalent: (A) the Hausdorff and box dimensions of K coincide; (B) with respect to some gauge function, the Hausdorff measure of K is positive and finite; (C) the Hausdorff dimension of the measure of maximal entropy on K attains the Hausdorff dimension of K . When $s \geq 3$, we find some examples in which (A) does not hold but (C) holds, which is a new phenomenon not appearing in the planar cases. Through a different probabilistic approach, we establish the equivalence of (A) and (B) for Bedford-McMullen sponges.

1. INTRODUCTION

The Hausdorff and box dimensions are the most common notions to quantify the size of sets in fractal geometry; see [15, 31] for an introduction. It is natural to ask under which condition these two dimensions coincide, especially for invariant sets in dynamical systems. This is an important question and has a long history. Furstenberg [20] showed that if K is a compact set invariant under an expanding conformal endomorphism on the d -torus, then

$$(A) \quad \dim_H K = \dim_B K,$$

where \dim_H and \dim_B stand for the Hausdorff and box dimensions respectively. Subsequently, (A) has been proved to hold in various conformal settings [3, 13, 22] and some typical nonconformal settings [12, 24]. Notably, Falconer proved that (A) holds for all self-similar sets [13] and certain typical self-affine sets [12]. In contrast, the Hausdorff and box dimensions are usually distinct in specific nonconformal settings. In this paper, we study when (A) holds for the invariant sets in the following family of nonconformal dynamical systems. Let T be an expanding linear endomorphism on the d -torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ represented by a diagonal integer matrix

$$\Lambda = \text{diag}(m_1, \dots, m_d),$$

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where $m_1 \geq \dots \geq m_d \geq 2$, that is, $T(x) = \Lambda x \mod 1$ for $x \in [0, 1)^d$. Write the number of distinct expanding ratios as

$$(1.1) \quad s = \#\{m_i : 1 \leq i \leq d\},$$

where $\#$ stands for cardinality. Let $\mathcal{A} = \prod_{j=1}^d \{0, \dots, m_j - 1\}$. There is a canonical representation map $R: \mathcal{A}^{\mathbb{N}} \rightarrow \mathbb{T}^d$ given by

$$(1.2) \quad R(x) = \sum_{k=1}^{\infty} \Lambda^{-k} x_k \quad \text{for } x = (x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}.$$

Let σ be the (left) shift on $\mathcal{A}^{\mathbb{N}}$ defined as $\sigma((x_k)_{k=1}^{\infty}) = (x_{k+1})_{k=1}^{\infty}$. Endow $\mathcal{A}^{\mathbb{N}}$ with the product topology. A closed subset X of $\mathcal{A}^{\mathbb{N}}$ is called a *subshift* if $\sigma(X) \subset X$. Since R is a continuous surjective map and $R \circ \sigma = T \circ R$, there is an one-to-one correspondence between the subshifts of $\mathcal{A}^{\mathbb{N}}$ and the compact T -invariant subsets of \mathbb{T}^d . When $K = R(\mathcal{D}^{\mathbb{N}})$ for some nonempty subset \mathcal{D} of \mathcal{A} , we call K a *Bedford-McMullen carpet* if $d = 2$ and a *Bedford-McMullen sponge* if $d \geq 3$.

Given a continuous increasing function $\varphi: [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$, called a *gauge function*, for $E \subset \mathbb{R}^d$ and $\delta > 0$, define

$$\mathcal{H}_{\delta}^{\varphi}(E) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(|E_i|) : \{E_i\}_{i=1}^{\infty} \text{ is a cover of } E \text{ and } |E_i| \leq \delta \text{ for all } i \right\},$$

where $|F|$ denotes the diameter of a set $F \subset \mathbb{R}^d$ in Euclidean norm. The *Hausdorff measure* of E is

$$\mathcal{H}^{\varphi}(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_{\delta}^{\varphi}(E).$$

Write $\mathcal{H}_{\delta}^{\varphi}$ as $\mathcal{H}_{\delta}^{\gamma}$ and \mathcal{H}^{φ} as \mathcal{H}^{γ} if $\varphi(t) = t^{\gamma}$ for some $\gamma \geq 0$. We say that there exists a gauge function for $K \subset \mathbb{R}^d$ if

$$(B) \quad 0 < \mathcal{H}^{\varphi}(K) < \infty \quad \text{for some gauge function } \varphi.$$

In this paper, for a compact T -invariant set K , we aim to characterize the coincidence of dimensions as in (A) by the existence of gauge functions as in (B).

Our research target is motivated as follows. In 1980s, Bedford [4] and McMullen [32] independently computed the Hausdorff and box dimensions of Bedford-McMullen carpets. Later their work was extended by Kenyon and Peres [25, 26] to Bedford-McMullen sponges and certain sofic affine-invariant sets; see [1, 18, 19, 29] for some other generalizations. For a Bedford-McMullen carpet K , McMullen [32] pointed out that $0 < \mathcal{H}^{\dim_{\text{H}} K}(K) < \infty$ if $\dim_{\text{H}} K = \dim_{\text{B}} K$, and asked about the value of $\mathcal{H}^{\dim_{\text{H}} K}(K)$ when $\dim_{\text{H}} K < \dim_{\text{B}} K$. This question was answered by Peres [37] who proved that $\mathcal{H}^{\dim_{\text{H}} K}(K) = \infty$ if $\dim_{\text{H}} K < \dim_{\text{B}} K$. Afterwards, Peres [36] showed that if $\dim_{\text{H}} K < \dim_{\text{B}} K$, the packing measure of K at its packing dimension (which coincides with $\dim_{\text{B}} K$, see e.g. [15, Corollary 3.10]) is also infinite. Before the result of Peres [37], an elegant argument found independently by Bedford and Mandelbrot (unpublished) and by Gatzouras and Lalley [29] gives the equivalence of (A) and (B) for Bedford-McMullen carpets (see e.g. [37, Proposition 2]). Actually, Gatzouras and Lalley [29] showed that $\dim_{\text{H}} K = \dim_{\text{B}} K$ is equivalent to $0 < \mathcal{H}^{\dim_{\text{H}} K}(K) < \infty$ for a *Gatzouras-Lalley carpet* K which is a generalization of

Bedford-McMullen carpet by relaxing the grid structure but keeping the dominated directions. These previous results motivate us to study the equivalence of (A) and (B) for invariant subsets of Bedford-McMullen sponges (also called *sub-self-affine sets* [14, 24]).

To state our results, we introduce another related condition that the Hausdorff dimension of the measure of maximal entropy attains the Hausdorff dimension of its support. For a subshift X of $\mathcal{A}^{\mathbb{N}}$, write

$$\mathcal{L}_k(X) = \{i_1 \cdots i_k \in \mathcal{A}^k : i_1 \cdots i_k = x_1 \cdots x_k \text{ for some } (x_j)_{j=1}^{\infty} \in X\}, \quad k \in \mathbb{N},$$

and

$$\mathcal{L}(X) = \bigcup_{k=1}^{\infty} \mathcal{L}_k(X) \bigcup \{\emptyset\},$$

where \emptyset denotes the empty word. For $I \in \mathcal{A}^k$, denote $|I| = k$. Set $|\emptyset| = 0$. Let IJ denote the juxtaposition of $I, J \in \mathcal{L}(X)$. We say a subshift X satisfies *weak specification* if there exists an integer p such that for every $I, J \in \mathcal{L}(X)$, there is $W \in \mathcal{L}(X)$ with $|W| \leq p$ such that $IWJ \in \mathcal{L}(X)$. The concept of specification traces back to the seminal work of Bowen [5, 6] on the uniqueness of equilibrium states (see [8]), and plays a key role in the existence of the box dimension of $R(X)$ (see Proposition 3.3 and [23]). Let X be a subshift satisfying weak specification and $K = R(X)$. There is a unique σ -invariant *measure of maximal entropy* μ on X (see Proposition 2.2). Then we give another condition that

$$(C) \quad \dim_{\mathrm{H}} R\mu = \dim_{\mathrm{H}} K,$$

where $R\mu = \mu \circ R^{-1}$ is the push-forward of μ under R , and $\dim_{\mathrm{H}} R\mu$ is the Hausdorff dimension of $R\mu$ defined to be the infimum of the Hausdorff dimension of Borel sets with positive $R\mu$ -measure.

Now we are ready to state our first result.

Theorem 1.1. *Let X be a subshift satisfying weak specification and $K = R(X)$. Recall the definition of s from (1.1). Then the following statements hold.*

- (i) (A) implies $0 < \mathcal{H}^{\dim_{\mathrm{H}} K}(K) < \infty$, and so (B).
- (ii) (B) implies (C).
- (iii) If $s \leq 2$, then (A), (B), (C) are equivalent.
- (iv) If $s \geq 3$, there are Bedford-McMullen sponges in which (A) does not hold but (C) holds.

In [37, P. 526], Peres conjectured that when $d = 2$ and $K = R(X)$ where X is a primitive subshift of finite type, $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) = \infty$ if $\dim_{\mathrm{H}} K < \dim_{\mathrm{B}} K$. Since such X satisfies weak specification, it follows from Theorem 1.1(iii) that $\mathcal{H}^{\dim_{\mathrm{H}} K}(K)$ is either 0 or ∞ if $\dim_{\mathrm{H}} K < \dim_{\mathrm{B}} K$.

Let us give some ideas about the proof of Theorem 1.1. Firstly, we characterize (A) in terms of the size of the fibers of certain factor maps; see Theorem 3.1(c), which is by adapting and extending some arguments of [25] to the setting of weak specification. Based on this characterization and the Gibbs property of the measure of maximal entropy, we obtain an upper bound on the density of $R\mu$, leading to the part of (i) that $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) >$

0. Unlike Bedford-McMullen carpets corresponding to the full shifts, in the case of subshifts it is difficult to estimate the density of $R\mu$ from below. Instead of following the approach based on the density to prove the other part of (i) that $\mathcal{H}^{\dim_{\mathbb{H}} K}(K) < \infty$, we directly consider the covers consisting of the approximate cubes whose number is estimated using weak specification. The proof of (ii) is based on the translation-invariance of Hausdorff measures and the Gibbs property of the measure of maximal entropy; see Proposition 5.2. Next we move to the proof of (iii) which is more difficult and requires new ideas since the Hausdorff dimension of T -invariant sets is expressed as a certain topological pressure instead of the simple formula for Bedford-McMullen carpets. By (i) and (ii), it suffices to show that (C) implies (A) when $s \leq 2$; see Proposition 5.3. To this end, we manage to compute the Hausdorff dimension of the invariant set based on the key observation that the projection of the measure of full dimension to the weakest unstable direction always has the Gibbs property. As for (iv), we give some examples of Bedford-McMullen sponges (Example 7.1 and Example 7.2) with the desired properties, which is a new phenomenon not appearing in the planar cases.

To determine the Hausdorff measures of Example 7.1 and Example 7.2, we manage to give a sufficient condition for Bedford-McMullen sponges to have infinite Hausdorff measures (see Proposition 6.6), and conclude that Example 7.1 and Example 7.2 have infinite Hausdorff measures at their respective Hausdorff dimensions. The proof of Proposition 6.6 is based on the method of Peres [37] (see Proposition 6.5) and a newfound relation satisfied by the measure of full dimension (see Lemma 6.8). It is worth pointing out that in [37, Section 5] Peres mentioned the result ([37, Theorem 1]), that $\mathcal{H}^{\dim_{\mathbb{H}} K}(K) = \infty$ if $\dim_{\mathbb{H}} K < \dim_{\mathbb{B}} K$ for a Bedford-McMullen carpet K , can extend to Bedford-McMullen sponges. However, he did not give a detailed justification.

The equivalence of (A) and (B) may still hold when $s \geq 3$. However, (iv) shows that our current strategy for the proof of (iii) can not extend to $s \geq 3$. Nevertheless, through a different probabilistic approach we obtain the equivalence of (A) and (B) for Bedford-McMullen sponges.

Theorem 1.2. *(A) and (B) are equivalent for Bedford-McMullen sponges.*

The proof of Theorem 1.2 is based on the independence of the coordinates of a random sequence whose law is a Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$ and the density theorem for the Hausdorff measures. To the best of our knowledge, the previous known proofs for the equivalence of (A) and (B) for Bedford-McMullen carpets essentially rely on the equivalence of (A) and (C) (see e.g. [37, Proposition 2]), which does not work for Bedford-McMullen sponges as seen in Example 7.1. We remark that a similar proof of Theorem 1.2 based on the density theorem for the packing measures shows that for a Bedford-McMullen sponge K , $\dim_{\mathbb{H}} K = \dim_{\mathbb{B}} K$ is equivalent to that, with respect to some (doubling) gauge function φ , the packing measure of K is positive and finite. One may expect that Theorem 1.2 can extend to some subshifts if the limit theorem used in the proof of Theorem 1.2 holds under some weaker independence; see Remark 6.4 for an illustration.

The paper is organized as follows. In Section 2, we make some preparations for the proofs of main theorems. In Section 3, we extend the results in [25] to the setting of

weak specification. Section 4 is for the proof of Theorem 1.1(i). The proofs of (ii) and (iii) of Theorem 1.1 are given in Section 5. Section 6 is devoted to the results about Bedford-McMullen sponges including the proof of Theorem 1.2. The examples for Theorem 1.1(iv) are given in Section 7.

2. PRELIMINARIES

In Section 2.1, we introduce the subshifts and factor maps. Then we present some results about the measures of maximal entropy and full dimension respectively in Section 2.2 and Section 2.3. The last subsection is devoted to the approximate cubes and the related results.

Throughout this paper we shall mean by $a \lesssim b$ that $a \leq Cb$ for some positive constant C , and write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

2.1. Subshift and factor map. Let \mathcal{B} be a finite set. Let $k \in \mathbb{N}$ and $0 \leq a < b \leq k$. For $x = (x_i)_{i=1}^\infty \in \mathcal{B}^\mathbb{N}$, write $x|_a^b = x_{a+1} \dots x_b$ and $x|k = x_1 \dots x_k$. By convention we set $x|0 = \emptyset$ and $|\emptyset| = 0$, where \emptyset denotes the empty word. For $I = i_1 \dots i_k \in \mathcal{B}^k$, define $I|_a^b = i_{a+1} \dots i_b$, $|I| = k$, and $[I] = \{x \in \mathcal{B}^\mathbb{N} : x|k = I\}$. For $I \in \mathcal{B}^{k_1}$ and $J \in \mathcal{B}^{k_2}$, let IJ denote the juxtaposition of I and J . Endow $\mathcal{B}^\mathbb{N}$ with the product topology. Let σ be the shift on $\mathcal{B}^\mathbb{N}$ defined by $\sigma((x_k)_{k=1}^\infty) = (x_{k+1})_{k=1}^\infty$. A closed subset X of $\mathcal{B}^\mathbb{N}$ is called a *subshift* of $\mathcal{B}^\mathbb{N}$ if $\sigma(X) \subset X$. Let $\mathcal{M}_\sigma(X)$ denote the set of σ -invariant probability measures on X . Define

$$\mathcal{L}_k(X) = \{I \in \mathcal{B}^k : [I] \cap X \neq \emptyset\}, \quad k \in \mathbb{N} \quad \text{and} \quad \mathcal{L}(X) = \bigcup_{k=1}^\infty \mathcal{L}_k(X) \cup \{\emptyset\}.$$

Definition 2.1 (weak specification). A subshift X of $\mathcal{B}^\mathbb{N}$ is said to satisfy *weak specification* if there exists an integer p such that for every $I, J \in \mathcal{L}(X)$, there is $W \in \mathcal{L}(X)$ with $|W| \leq p$ such that $IWJ \in \mathcal{L}(X)$.

Recall $\mathcal{A} = \prod_{j=1}^d \{0, \dots, m_j - 1\}$ and $s = \#\{m_i : 1 \leq i \leq d\}$. Throughout this paper, a subshift refers by default to a subshift of $\mathcal{A}^\mathbb{N}$. Since $m_1 \geq \dots \geq m_d \geq 2$, there are integers $n_1 > \dots > n_s$ and $0 = d_0 < d_1 < \dots < d_s = d$ such that for $1 \leq i \leq s$

$$(2.1) \quad m_j = n_i, \quad d_{i-1} < j \leq d_i.$$

By convention we set $n_0 = \infty$ and $\mathcal{A}_0 = \mathcal{A}$. For $1 \leq i \leq s$, define

$$\mathcal{A}_i = \prod_{j > d_{i-1}} \{0, \dots, m_j - 1\}.$$

Let $\pi_1 : \mathcal{A} \rightarrow \mathcal{A}_1$ be the identity map. For $2 \leq i \leq s$, let $\pi_i : \mathcal{A}_{i-1} \rightarrow \mathcal{A}_i$ be the projection by removing first $(d_{i-1} - d_{i-2})$ coordinates, that is,

$$\pi_i((x_{d_{i-2}+1}, \dots, x_{d_{i-1}}, x_{d_{i-1}+1}, \dots, x_d)) = (x_{d_{i-1}+1}, \dots, x_d).$$

Naturally extend π_i from \mathcal{A}_{i-1} to $\mathcal{A}_{i-1}^\mathbb{N}$ by

$$\pi_i(x) = (\pi_i(x_k)) \quad \text{for } x = (x_k) \in \mathcal{A}_{i-1}^\mathbb{N}.$$

For $1 \leq i \leq s$, define $\tau_i: \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}_i^{\mathbb{N}}$ by $\tau_i = \pi_i \circ \cdots \circ \pi_1$. By abuse of notation we keep using σ to denote the shift on $\mathcal{A}_i^{\mathbb{N}}$ for $1 \leq i \leq s$, which shall not cause confusion since the domain of σ is always clear from the context. Then $\pi_i \circ \sigma = \sigma \circ \pi_i$ and $\tau_i \circ \sigma = \sigma \circ \tau_i$, which means π_i and τ_i are factor maps. For a subshift X and $1 \leq i \leq s$, define a subshift of $\mathcal{A}_i^{\mathbb{N}}$ by $X_i = \tau_i(X)$. Hence the following diagram commutes.

$$\begin{array}{ccccccc}
 & & & \tau_s & & & \\
 & & \nearrow & \tau_i & \searrow & & \\
 X & \xrightarrow{\pi_1} & X_1 & \xrightarrow{\pi_2} & \cdots & X_i & \cdots \xrightarrow{\pi_s} X_s
 \end{array}$$

For $1 \leq i \leq s$, by [Definition 2.1](#) and the surjectivity of τ_i , if X satisfies weak specification, then so does X_i .

2.2. Measure of maximal entropy. Let \mathcal{B} be a finite set and X be a subshift of $\mathcal{B}^{\mathbb{N}}$. The *topological entropy* $h(X)$ of X is

$$(2.2) \quad h(X) = \lim_{k \rightarrow \infty} \frac{1}{k} \log \# \mathcal{L}_k(X),$$

where the limit exists by a subadditivity argument. For $\mu \in \mathcal{M}_{\sigma}(X)$, the *measure-theoretic entropy* $h(\mu)$ of μ is

$$(2.3) \quad h(\mu) = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{I \in \mathcal{L}_k(X)} -\mu(I) \log \mu(I),$$

where the limit exists by the invariance of μ and a subadditivity argument. Note that $h(\mu) \leq h(X)$ since $\sum_{I \in \mathcal{L}_k(X)} -\mu(I) \log \mu(I) \leq \log \# \mathcal{L}_k(X)$ by the convexity. We call μ a *measure of maximal entropy* if $h(\mu) = h(X)$.

Suppose X satisfies weak specification. Applying [\[16, Theorem 5.5\]](#) with $\phi \equiv 1$ on X shows that there is a unique measure of maximal entropy with the Gibbs property, which is an extension of the classical result of Parry [\[35\]](#).

Proposition 2.2. *Let X be a subshift satisfying weak specification. Then there is a unique measure of maximal entropy μ on X . Moreover, μ is the unique ergodic measure satisfying the following Gibbs property*

$$\mu(I) \approx \frac{1}{\# \mathcal{L}_{|I|}(X)} \approx \exp(-|I| h(X)) \quad \text{for } I \in \mathcal{L}(X),$$

where $h(X)$ is as in [\(2.2\)](#).

2.3. Measure of full dimension. Roughly speaking, a measure of full dimension is an invariant measure having the same Hausdorff dimension as its support. It is an important topic in dynamical systems to determine the existence and uniqueness of the measure of full dimension on invariant sets [\[2, 16, 22, 26, 30, 33, 39\]](#); see [\[21\]](#) for a survey. In their celebrated work, Das and Simmons [\[9\]](#) constructed a self-affine sponge whose Hausdorff dimension is strictly greater than the supremum of the Hausdorff dimensions of its invariant measures, thus showing that the measure of full dimension may not exist in an expanding nonconformal system. For other dimensional properties of invariant measures

in nonconformal settings, very recently Kolossváry [28] computed the L^q dimensions of self-affine measures on self-affine sponges assuming a suitable separation condition, which generalizes the results in [27, 34].

For a subshift X and $K = R(X)$, we call an invariant measure μ on X a *measure of full dimension* if $\dim_{\text{H}} R\mu = \dim_{\text{H}} K$. Kenyon and Peres [26] established a Ledrappier-Young formula for the Hausdorff dimensions of ergodic measures and proved the existence of the measure of full dimension. For $\mu \in \mathcal{M}_{\sigma}(\mathcal{A}^{\mathbb{N}})$, following [26] we define

$$(2.4) \quad d_{\text{LY}}(\mu) = \sum_{i=1}^s \left(\frac{1}{\log n_i} - \frac{1}{\log n_{i-1}} \right) h(\tau_i \mu).$$

Proposition 2.3 ([26, Theorem 1.1 & Lemma 4.3]). *Let X be any subshift and $K = R(X)$. Then for each ergodic measure $\mu \in \mathcal{M}_{\sigma}(X)$,*

$$(2.5) \quad \dim_{\text{H}} R\mu = d_{\text{LY}}(\mu),$$

and

$$(2.6) \quad \dim_{\text{H}} K = \max_{\mu \in \mathcal{M}_{\sigma}(X)} d_{\text{LY}}(\mu).$$

By studying the optimization problem in (2.6), a part of the so-called *weighted thermodynamic formalism*, Feng [16] proved that when X satisfies weak specification, there is a unique measure of full dimension with certain lower Gibbs property. Before stating this result in Proposition 2.4, we need some notation. For $1 \leq i \leq s$, write

$$(2.7) \quad \alpha_i = \frac{\log n_i}{\log n_{i-1}}.$$

Let X be a subshift. Define $\phi^{(1)} \equiv 1$ on $\mathcal{L}(X)$. For $2 \leq i \leq s$, recursively define $\phi^{(i)}: \mathcal{L}(X_i) \rightarrow (0, \infty)$ by

$$(2.8) \quad \phi^{(i)}(J) = \sum_{I \in \mathcal{L}(X_{i-1}): \pi_i(I)=J} \phi^{(i-1)}(I)^{\alpha_{i-1}} \quad \text{for } J \in \mathcal{L}(X_i).$$

Then define $Z: \mathbb{N} \rightarrow (0, \infty)$ as

$$Z(k) = \sum_{I \in \mathcal{L}_k(X_s)} \phi^{(s)}(I)^{\alpha_s},$$

and

$$(2.9) \quad P = \lim_{k \rightarrow \infty} \frac{\log Z(k)}{k},$$

where the limit exists by a subadditivity argument.

Proposition 2.4 ([16, Theorems 5.5 & 7.3]). *Let X be a subshift satisfying weak specification and $K = R(X)$. Then there is a unique measure of full dimension η on X , that is,*

$$\dim_{\text{H}} R\eta = d_{\text{LY}}(\eta) = \max_{\mu \in \mathcal{M}_{\sigma}(X)} d_{\text{LY}}(\mu) = \dim_{\text{H}} K.$$

Moreover, η is ergodic and has the following lower Gibbs property,

$$(2.10) \quad \eta(I) \gtrsim \psi(I) \quad \text{for } I \in \mathcal{L}(X),$$

where

$$\psi(I) = \frac{1}{Z(|I|)} \prod_{i=1}^s \phi^{(i)}(\tau_i(I))^{\alpha_i - 1}.$$

Additionally, $\tau_s \eta$ has the following Gibbs property,

$$(2.11) \quad \tau_s \eta(J) \approx \frac{\phi^{(s)}(J)^{\alpha_s}}{\exp(|J|P)} \quad \text{for } J \in \mathcal{L}(X_s),$$

where P is as in (2.9).

Combining Proposition 2.3 and [16, Proposition 3.7] (a relative variational principle for subadditive potentials) gives a formula for the Hausdorff dimension of all compact T -invariant sets, an analog of which for certain sofic affine-invariant sets was previously established in [25, Theorem 1.1].

Lemma 2.5. *Let X be a subshift and $K = R(X)$. Then*

$$(2.12) \quad \dim_{\text{H}} K = \frac{P}{\log n_s}$$

where P is as in (2.9).

2.4. Approximate cube. In this subsection, we introduce the approximate cubes with side lengths $\approx n_s^{-k}$ ($k \in \mathbb{N}$) by “slowing down” the iterations along the stronger unstable directions. For $1 \leq i \leq s$, write

$$(2.13) \quad \theta_i = \frac{\log n_s}{\log n_i}.$$

For $k \in \mathbb{N}$, define $R_k: \bigcup_{n \geq k} \mathcal{A}^n \rightarrow \mathbb{T}^d$ by

$$(2.14) \quad R_k(I) := \sum_{\ell=1}^k \Lambda^{-\ell} i_{\ell} \quad \text{for } I = i_1 \cdots i_k \cdots i_n.$$

Combining (1.2) and (2.14) gives

$$(2.15) \quad R(x) = R_k(x|k) + \Lambda^{-k} R(\sigma^k x) \quad \text{for } x \in \mathcal{A}^{\mathbb{N}}.$$

Let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. For $I \in \mathcal{A}^k$, the k -level *approximate cube* is

$$(2.16) \quad Q_k(I) := \prod_{i=1}^s \prod_{j=d_{i-1}+1}^{d_i} \left[R_{\lfloor \theta_i k \rfloor}(I)_j, R_{\lfloor \theta_i k \rfloor}(I)_j + n_i^{-\lfloor \theta_i k \rfloor} \right],$$

where v_j denotes the j -th coordinate of a vector $v \in \mathbb{R}^d$. For $x \in \mathcal{A}^{\mathbb{N}}$, we define $Q_k(x) = Q_k(x|k)$. Combining (2.1), (2.13) and (2.16) shows that $|Q_k(x)| \approx n_s^{-k}$.

We begin with a lemma relating the measure of approximate cubes to that of a collection of cylinders in the symbolic space. For $x \in X$ and $k \in \mathbb{N}$, define

$$(2.17) \quad \Gamma_k(x) = \left\{ I \in \mathcal{L}_k(X) : \tau_i(I|[\theta_i k]) = \tau_i(x|[\theta_i k]) \text{ for } 1 \leq i \leq s \right\}.$$

Lemma 2.6. *Let X be a subshift and μ be a Borel measure on X . Then for $x \in X$,*

$$(2.18) \quad R\mu(Q_k(x)) = \sum_{I \in \Gamma_k(x)} \mu(I).$$

Proof. Let $K = R(X)$. It follows from (2.16) and (2.17) that

$$R^{-1}(Q_k(x) \cap K) \subset \bigcup_{I \in \Gamma_k(x)} [I]$$

and

$$R^{-1}(Q_k(x) \cap K) \cap [I] \neq \emptyset \quad \text{for } I \in \Gamma_k(x).$$

Since $R\mu$ is supported on K ,

$$R\mu(Q_k(x)) = R\mu(Q_k(x) \cap K) = \sum_{I \in \Gamma_k(x)} \mu(I),$$

which finishes the proof. \square

Next we give a lemma for counting the number of all k -level approximate cubes. Denote

$$(2.19) \quad \mathcal{Q}_k(X) = \{Q_k(x) : x \in X\}.$$

Then $\mathcal{Q}_k(X)$ covers K by the k -level approximate cubes. To estimate $\#\mathcal{Q}_k(X)$, following [32] we introduce a collection of vectors

$$(2.20) \quad \mathcal{D}_k(X) := \left\{ (J_1, \dots, J_s) \in \prod_{i=1}^s \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i) : \text{there exists } I \in \mathcal{L}_k(X) \right. \\ \left. \text{such that } \tau_i(I|_{[\theta_{i-1} k]^{[\theta_i k]}}) = J_i \text{ for } 1 \leq i \leq s \right\}.$$

Lemma 2.7. *Let X be a subshift. Then $\#\mathcal{Q}_k(X) = \#\mathcal{D}_k(X)$ for $k \in \mathbb{N}$.*

Proof. It suffices to construct a bijective map $f: \mathcal{Q}_k(X) \rightarrow \mathcal{D}_k(X)$. For $Q \in \mathcal{Q}_k(X)$, there exists $x \in X$ such that $Q = Q_k(x)$. Define

$$f(Q) := (\tau_1(x|_{[\theta_{-1} k]^{[\theta_1 k]}}), \dots, \tau_s(x|_{[\theta_{s-1} k]^{[\theta_s k]}})).$$

Then f is injective since for $x, y \in X$, $Q_k(x) = Q_k(y)$ if and only if $\tau_i(x|_{[\theta_{i-1} k]^{[\theta_i k]}}) = \tau_i(y|_{[\theta_{i-1} k]^{[\theta_i k]}})$ for $1 \leq i \leq s$. Next we show that f is surjective. For $\mathbf{J} = (J_1, \dots, J_s) \in \mathcal{D}_k(X)$, by (2.20) there is $I \in \mathcal{L}_k(X)$ such that $\tau_i(I|_{[\theta_{i-1} k]^{[\theta_i k]}}) = J_i$ for $1 \leq i \leq s$. Take some $x \in [I] \cap X$, then $f(Q(x)) = \mathbf{J}$. \square

An upper bound on the upper box dimension of $R(X)$ follows immediately.

Lemma 2.8. *Let X be a subshift and $K = R(X)$. Then*

$$\overline{\dim}_B K \leq \sum_{i=1}^s \left(\frac{1}{\log n_i} - \frac{1}{\log n_{i-1}} \right) h(X_i).$$

Proof. By definition of $\mathcal{D}_k(X)$,

$$(2.21) \quad \log \# \mathcal{D}_k(X) \leq \log \# \left(\prod_{i=1}^s \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i) \right) \leq \sum_{i=1}^s \log \# \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i).$$

For $1 \leq i \leq s$, since $h(X_i) = \lim_{k \rightarrow \infty} (1/k) \log \# \mathcal{L}_k(X_i)$ and $\theta_i - \theta_{i-1} > 0$,

$$(2.22) \quad \lim_{k \rightarrow \infty} \frac{\log \# \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i)}{k} = \lim_{k \rightarrow \infty} \frac{[\theta_i k] - [\theta_{i-1} k]}{k} \cdot \frac{\log \# \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i)}{[\theta_i k] - [\theta_{i-1} k]} = (\theta_i - \theta_{i-1}) h(X_i).$$

Then

$$\begin{aligned} \overline{\dim}_B K &\leq \limsup_{k \rightarrow \infty} \frac{\log \# \mathcal{Q}_k(X)}{\log n_s^k} \\ &= \limsup_{k \rightarrow \infty} \frac{\log \# \mathcal{D}_k(X)}{\log n_s^k} && \text{(by Lemma 2.7)} \\ &\leq \frac{1}{\log n_s} \sum_{i=1}^s \limsup_{k \rightarrow \infty} \frac{\log \# \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i)}{k} && \text{(by (2.21))} \\ &= \frac{1}{\log n_s} \left(\sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i) \right) && \text{(by (2.22)),} \end{aligned}$$

which finishes the proof. \square

We end this section with following variant of the Rogers-Taylor theorem [40] which is a useful tool to estimate the Hausdorff measure.

Lemma 2.9. *Let $E \subset \mathbb{T}^d$ be a Borel set and ν be a Borel measure with $0 < \nu(E) < \infty$. Then for a gauge function φ ,*

- (i) *if $\limsup_{k \rightarrow \infty} \nu(Q_k(x))/\varphi(n_s^{-k}) \leq C$ for ν -a.e. $x \in E$, then $\mathcal{H}^\varphi(E) \gtrsim \nu(E)/C$.*
- (ii) *if $\limsup_{k \rightarrow \infty} \nu(Q_k(x))/\varphi(n_s^{-k}) \geq C$ for all $x \in E$, then $\mathcal{H}^\varphi(E) \lesssim \nu(E)/C$.*

3. SOME CHARACTERIZATIONS FOR THE COINCIDENCE OF DIMENSIONS

In this section we prove the following theorem which provides some dynamical characterizations of the coincidence of the Hausdorff and box dimensions for $K = R(X)$ when X is a subshift satisfying weak specification.

Theorem 3.1. *Let X be a subshift satisfying weak specification and $K = R(X)$. For $1 \leq i \leq s$, let μ_i be the measure of maximal entropy on X_i (see Proposition 2.2). Then the following statements are equivalent.*

- (a) $\dim_H K = \dim_B K$.
- (b) $\tau_i \mu_1 = \mu_i$ for $1 \leq i \leq s$.
- (c) *For $1 \leq i \leq s$ and $I \in \mathcal{L}(X_i)$, $\# \tau_i^{-1}(I) \approx \exp(|I|(h(X_1) - h(X_i)))$, where $h(X_i)$ is as in (2.2).*
- (d) *There exist $0 \leq \lambda_1 \leq \dots \leq \lambda_s$ such that for $1 \leq i \leq s$ and $I \in \mathcal{L}(X_i)$, $\# \tau_i^{-1}(I) \approx \exp(\lambda_i |I|)$.*

Let us provide some explanations for the above statements. (b) asserts that measures of maximal entropy should be projected to measures of maximal entropy along the chain of factor maps $X_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} X_s$. (d) states that fibers over words of the same length have comparable sizes, while (c) gives a more precise expression of this size in terms of topological entropy. When $X = \mathcal{D}^{\mathbb{N}}$ is a full shift, (d) can be verified by directly checking the fibers over $\tau_i(\mathcal{D})$ for $1 \leq i \leq s$. In the case where X is an irreducible subshift of finite type, Feng, Lo, and Shen [17, Theorem 1.7] provided an algorithm to determine whether (d) holds. The proof of Theorem 3.1 adapts and extends ideas from [25].

3.1. The equivalence of (a) and (b). We first compute the box dimension, which relies on the next lemma about the number of approximate cubes.

Lemma 3.2. *Let X be a subshift satisfying weak specification. Then for $k \in \mathbb{N}$,*

$$\#\mathcal{Q}_k(X) \approx \exp \left(k \sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i) \right).$$

Proof. For $1 \leq i \leq s$, applying Proposition 2.2 to X_i gives

$$(3.1) \quad \#\mathcal{L}_k(X_i) \approx \exp(kh(X_i)).$$

By Lemma 2.7 and (2.20),

$$\#\mathcal{Q}_k(X) = \#\mathcal{D}_k(X) \leq \prod_{i=1}^s \#\mathcal{L}_{[\theta_i k] - [\theta_{i-1} k]}(X_i) \approx \exp \left(k \sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i) \right).$$

Next we control $\#\mathcal{Q}_k(X)$ from below. Let p be as in Definition 2.1. For $k \in \mathbb{N}$ large, define

$$(3.2) \quad \mathcal{J}_k = \prod_{i=1}^s \mathcal{L}_{[\theta_i k] - [\theta_{i-1} k] - p}(X_i),$$

and

$$\mathcal{W} = \left\{ \prod_{i=1}^s (W_i^l, W_i^r) \in \prod_{i=1}^s \mathcal{L}(X) \times \mathcal{L}(X) : |W_i^l| + |W_i^r| = p \right\}.$$

We proceed to construct a map g from a subset of $\mathcal{J}_k \times \mathcal{W}$ to $\mathcal{D}_k(X)$. For $\mathbf{J} = (J_1, \dots, J_s) \in \mathcal{J}_k$, by the surjectivity of τ_i there exists $\mathbf{I} = (I_1, \dots, I_s) \in \mathcal{L}(X)^s$ such that $\tau_i(I_i) = J_i$ for $1 \leq i \leq s$. Then we define W_i^l and W_i^r recursively for $1 \leq i \leq s$. Take $W_1^l = \emptyset$ and $W_1^r \in \mathcal{L}_p(X)$ such that $I_1 W_1^r \in \mathcal{L}(X)$. Suppose for some $1 \leq i \leq s-1$ we have found $W_1^l, W_1^r, \dots, W_i^l, W_i^r$ such that

$$W_1^l I_1 W_1^r \cdots W_i^l I_i W_i^r \in \mathcal{L}(X) \quad \text{and} \quad |W_j^l| + |W_j^r| = p \quad \text{for } 1 \leq j \leq i.$$

By Definition 2.1, there exists $W_{i+1}^l \in \mathcal{L}(X)$ with $|W_{i+1}^l| \leq p$ such that $W_i^r W_{i+1}^l I_{i+1} \in \mathcal{L}(X)$. Take $W_{i+1}^r \in \mathcal{L}_{p-|W_{i+1}^l|}(X)$ such that $I_{i+1} W_{i+1}^r \in \mathcal{L}(X)$. Hence we find $\mathbf{W} = (W_1^l, W_1^r, \dots, W_s^l, W_s^r)$ such that

$$(3.3) \quad W_1^l I_1 W_1^r \cdots W_s^l I_s W_s^r \in \mathcal{L}(X) \quad \text{and} \quad |W_i^l| + |W_i^r| = p \quad \text{for } 1 \leq i \leq s.$$

Thus, for $\mathbf{J} \in \mathcal{J}_k$ we find some $\mathbf{W} \in \mathcal{W}$ as above and define

$$g(\mathbf{J}, \mathbf{W}) = \prod_{i=1}^s \tau_i(W_i^l I_i W_i^r).$$

Denote the domain and range of g by $\text{Dom}(g)$ and $\text{Ran}(g)$ respectively. Since g is defined for all $\mathbf{J} \in \mathcal{J}_k$, we have $\#\text{Dom}(g) \geq \#\mathcal{J}_k$. By (3.3) and (2.20), $\text{Dom}(g) \subset \mathcal{J}_k \times \mathcal{W}$ and $\text{Ran}(g) \subset \mathcal{D}_k(X)$. Note that $\#g^{-1}(\mathbf{U}) \leq \#\mathcal{W}$ for $\mathbf{U} \in \text{Ran}(g)$ by the definition of g . Hence

$$\begin{aligned} \#\mathcal{Q}_k(X) &= \#\mathcal{D}_k(X) && \text{(by Lemma 2.7)} \\ &\geq \#\text{Ran}(g) && \text{(by } \text{Ran}(g) \subset \mathcal{D}_k(X)) \\ &\geq \frac{\#\text{Dom}(g)}{\#\mathcal{W}} && \text{(by } \#g^{-1}(\mathbf{U}) \leq \#\mathcal{W}, \mathbf{U} \in \text{Ran}(g)) \\ &\geq \frac{\#\mathcal{J}_k}{\#\mathcal{W}} && \text{(by } \#\text{Dom}(g) \geq \#\mathcal{J}_k) \\ &\gtrsim \prod_{i=1}^s \#\mathcal{L}_{[\theta_i k] - [\theta_{i-1} k] - p}(X_i) && \text{(by (3.2) and } \#\mathcal{W} \leq (\#\mathcal{A})^{2p}) \\ &\approx \exp\left(k \sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i)\right) && \text{(by (3.1)),} \end{aligned}$$

for k large enough depending on p . \square

Since $\mathcal{Q}_k(X)$ covers K by approximate cubes with side length $\approx n_s^{-k}$, Lemma 3.2 implies the following formula for the box dimension of K .

Proposition 3.3. *Let X be a subshift satisfying weak specification and $K = R(X)$. Then*

$$(3.4) \quad \dim_{\text{B}} K = \sum_{i=1}^s \left(\frac{1}{\log n_i} - \frac{1}{\log n_{i-1}} \right) h(X_i)$$

where $h(X_i)$ is as in (2.2).

It is worth pointing out that recently Jurga [23] showed that the box dimension of $K = R(X)$ may not exist if X does not satisfy weak specification.

The equivalence of (a) and (b) follows immediately.

Proof of (a) \iff (b). By Proposition 2.3, let η be an ergodic measure on X such that $d_{\text{LY}}(\eta) = \dim_{\text{H}} K$. By (2.4) and Proposition 3.3, (a) is equivalent to

$$(3.5) \quad \sum_{i=1}^s \left(\frac{1}{n_i} - \frac{1}{n_{i-1}} \right) (h(X_i) - h(\tau_i \eta)) = 0.$$

Since $h(\tau_i \eta) \leq h(X_i)$ and $n_i < n_{i-1}$ for $1 \leq i \leq s$, (3.5) holds if and only if

$$h(X_i) = h(\tau_i \eta) \quad \text{for } 1 \leq i \leq s.$$

The proof is finished by Proposition 2.2. \square

3.2. The equivalence of (b), (c) and (d).

Proof of the equivalence of (b), (c) and (d). By Proposition 2.2, for $1 \leq i \leq s$ and $I \in \mathcal{L}_k(X_i)$,

$$(3.6) \quad \mu_i(I) \approx \frac{1}{\#\mathcal{L}_k(X_i)} \approx \exp(-kh(X_i)).$$

Fix any $i \in \{1, \dots, s\}$ and $k \in \mathbb{N}$.

If (b) holds, then for $I \in \mathcal{L}_k(X_i)$,

$$\exp(-kh(X_i)) \approx \mu_i(I) = \tau_i \mu_1(I) = \sum_{J \in \tau_i^{-1}(I)} \mu(J) \approx \#\tau_i^{-1}(I) \cdot \exp(-kh(X_1)).$$

This implies (c).

It is immediate that (d) follows from (c) by taking $\lambda_i = h(X_1) - h(X_i)$.

Next we show that (d) implies (b). By (3.6) and (d), for $I \in \mathcal{L}_k(X_i)$,

$$(3.7) \quad \tau_i \mu_1(I) = \sum_{J \in \tau_i^{-1}(I)} \mu(J) \approx \#\tau_i^{-1}(I) \cdot \exp(-kh(X_1)) \approx \exp(k(\lambda_i - h(X_1))).$$

Then

$$\#\mathcal{L}_k(X_i) \cdot \exp(k(\lambda_i - h(X_1))) \approx \sum_{I \in \mathcal{L}_k(X_i)} \tau_i \mu_1(I) = 1.$$

Since $\#\mathcal{L}_k(X_i) \approx \exp(kh(X_i))$ by (3.6), we have $\lambda_i = h(X_1) - h(X_i)$. Combining (3.7) and (3.6) gives

$$\tau_i \mu_1(I) \approx \exp(-kh(X_i)) \approx \mu_i(I).$$

This shows (b) by Proposition 2.2. □

4. PROOF OF THEOREM 1.1(I)

This section is devoted to the proof of Theorem 1.1(i). We begin with a lemma controlling the number of cylinders in the approximate cube by topological entropies.

Lemma 4.1. *Let X be a subshift satisfying weak specification and $K = R(X)$. If $\dim_{\mathbb{H}} K = \dim_{\mathbb{B}} K$, then for $x \in X$ and $k \in \mathbb{N}$,*

$$(4.1) \quad \#\Gamma_k(x) \lesssim \exp\left(k \sum_{i=1}^s (\theta_i - \theta_{i-1})(h(X) - h(X_i))\right),$$

where $\Gamma_k(\cdot)$ is as in (2.17).

Proof. For $1 \leq i \leq j \leq s$ and $I, J \in \mathcal{L}(X)$, $\tau_i(I) = \tau_i(J)$ implies $\tau_j(I) = \tau_j(J)$ since $\tau_j = \pi_j \circ \dots \circ \pi_{i+1} \circ \tau_i$. Then by (2.17),

$$\begin{aligned} \#\Gamma_k(x) &= \#\left\{I \in \mathcal{L}_k(X) : \tau_i(I|[\theta_i k]) = \tau_i(x|[\theta_i k]) \text{ for } 1 \leq i \leq s\right\} \\ &= \#\left\{I \in \mathcal{L}_k(X) : \tau_i(I|[\theta_i k]_{[\theta_{i-1} k]}) = \tau_i(x|[\theta_i k]_{[\theta_{i-1} k]}) \text{ for } 1 \leq i \leq s\right\} \end{aligned}$$

$$\begin{aligned}
&\leq \prod_{i=1}^s \# \tau_i^{-1} \left(\tau_i(x|_{[\theta_{i-1}k]^{[\theta_i k]}}) \right) \\
&\approx \prod_{i=1}^s \exp \left(k(\theta_i - \theta_{i-1})(h(X) - h(X_i)) \right),
\end{aligned}$$

where the last equality is by (c) of Theorem 3.1. \square

Now we are ready to prove Theorem 1.1(i).

Proof of Theorem 1.1(i). Suppose $\dim_H K = \dim_B K = \gamma$. Then Proposition 3.3 shows

$$(4.2) \quad \gamma = \frac{1}{\log n_s} \sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i).$$

We first show $\mathcal{H}^\gamma(K) > 0$. By Proposition 2.2, let μ be the unique measure of maximal entropy on X . Then

$$(4.3) \quad \mu(I) \approx \exp(-kh(X)) \quad \text{for } I \in \mathcal{L}_k(X).$$

For $x \in X$ and $k \in \mathbb{N}$,

$$\begin{aligned}
R\mu(Q_k(x)) &= \sum_{I \in \Gamma_k(x)} \mu(I) && \text{(by Lemma 2.6)} \\
&\approx \exp(-kh(X)) \cdot \#\Gamma_k(x) && \text{(by (4.3))} \\
&\lesssim \exp \left(-k \left(h(X) - \sum_{i=1}^s (\theta_i - \theta_{i-1})(h(X) - h(X_i)) \right) \right) && \text{(by Lemma 4.1)} \\
&= \exp \left(-k \left(\sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i) \right) \right) && \text{(by } \theta_s = 1, \theta_0 = 0) \\
&= n_s^{-k\gamma} && \text{(by (4.2)).}
\end{aligned}$$

This shows $\mathcal{H}^\gamma(K) \gtrsim 1$ by Lemma 2.9.

Next we prove $\mathcal{H}^s(K) < \infty$. Since $\mathcal{Q}_k(X)$ covers K by the k -level approximate cubes with side lengths $\approx n_s^{-k}$, Lemma 3.2 implies that for some $C > 0$,

$$\mathcal{H}_{Cn_s^{-k}}^\gamma(K) \lesssim n_s^{-k\gamma} \cdot \#\mathcal{Q}_k(X) \approx n_s^{-k\gamma} \exp \left(k \sum_{i=1}^s (\theta_i - \theta_{i-1}) h(X_i) \right) = 1,$$

where the last equality is by (4.2). Letting $k \rightarrow \infty$ gives $\mathcal{H}^\gamma(K) \lesssim 1$. \square

5. PROOF OF (II) AND (III) IN THEOREM 1.1

Theorem 1.1(ii) is contained in Proposition 5.2. Theorem 1.1(iii) follows by combining Proposition 5.3 with (i) and (ii) of Theorem 1.1.

5.1. Proof of Theorem 1.1(ii). We first show that the Hausdorff measures of R -images of cylinders with the same length are comparable to each other.

Lemma 5.1. *Let X be a subshift satisfying weak specification and $K = R(X)$. Let φ be a gauge function. Then for $k \in \mathbb{N}$ and $I \in \mathcal{L}_k(X)$,*

$$\mathcal{H}^\varphi(R([I]) \cap K) \approx \mathcal{H}^\varphi(\Lambda^{-k}K).$$

Proof. For $J \in \mathcal{L}(X)$, write $\mathcal{D}_J = \{j \in \mathcal{A} : Jj \in \mathcal{L}(X)\}$. By the translation-invariance of \mathcal{H}^φ and (2.15),

$$(5.1) \quad \mathcal{H}^\varphi(R([J] \cap X)) = \sum_{j \in \mathcal{D}_J} \mathcal{H}^\varphi(R([Jj] \cap X)) = \sum_{j \in \mathcal{D}_J} \mathcal{H}^\varphi(\Lambda^{-|J|}R([j] \cap X)).$$

Then since $\mathcal{D}_I \subset \mathcal{L}_1(X)$, $X = \bigcup_{j \in \mathcal{L}_1(X)} [j] \cap X$, and $K = R(X)$,

$$(5.2) \quad \mathcal{H}^\varphi(R([I] \cap X)) \leq \sum_{j \in \mathcal{L}_1(X)} \mathcal{H}^\varphi(\Lambda^{-k}R([j] \cap X)) = \mathcal{H}^\varphi(\Lambda^{-k}R(X)) = \mathcal{H}^\varphi(\Lambda^{-k}K).$$

Next we give a lower bound on $\mathcal{H}^\varphi(R([I] \cap X))$. Let p be as in Definition 2.1. For $j \in \mathcal{L}_1(X)$, there exists $W \in \mathcal{L}(X)$ with $|W| \leq p$ such that $IWj \in \mathcal{L}(X)$. Then

$$(5.3) \quad \begin{aligned} \mathcal{H}^\varphi(R([I] \cap X)) &\geq \mathcal{H}^\varphi(R([IWj] \cap X)) \\ &\geq \mathcal{H}^\varphi(\Lambda^{-k-|W|}R([j] \cap X)) \quad (\text{by (5.1)}) \\ &\geq \mathcal{H}^\varphi(\Lambda^{-k-p}R([j] \cap X)), \end{aligned}$$

where the last inequality is by $|W| \leq p$ and Λ^{-1} is contracting. Summing (5.3) over $j \in \mathcal{L}_1(X)$ gives

$$(5.4) \quad \mathcal{H}^\varphi(R([I] \cap X)) \geq \frac{1}{\#\mathcal{A}} \mathcal{H}^\varphi(\Lambda^{-k-p}R(X)).$$

Notice that $\Lambda^{-k}T^{-p}(E) = \Lambda^{-k-p}E + \Lambda^{-k} \sum_{J \in \mathcal{A}^p} R_p(J)$ for $E \subset \mathbb{T}^d$. By the translation-invariance of \mathcal{H}^φ ,

$$\mathcal{H}^\varphi(\Lambda^{-k-p}E) = \frac{1}{(\#\mathcal{A})^p} \mathcal{H}^\varphi(\Lambda^{-k}T^{-p}(E)).$$

Then by (5.4), taking $E = R(X)$ in the above equation implies that

$$(5.5) \quad \mathcal{H}^\varphi(R([I] \cap X)) \geq \frac{1}{(\#\mathcal{A})^{p+1}} \mathcal{H}^\varphi(\Lambda^{-k}T^{-p}(R(X))) \geq \frac{1}{(\#\mathcal{A})^{p+1}} \mathcal{H}^\varphi(\Lambda^{-k}K),$$

where the last inequality is by $K = R(X)$ and $K \subset T^{-p}(K)$. Combining (5.2) and (5.5) finishes the proof. \square

Now we are ready to prove Theorem 1.1(ii) which is contained in the following proposition.

Proposition 5.2. *Let X be a subshift satisfying weak specification and $K = R(X)$. Let μ be the measure of maximal entropy on X . If $0 < \mathcal{H}^\varphi(K) < \infty$ for some gauge function φ , then*

$$R\mu \approx \mathcal{H}^\varphi|_K$$

where $\mathcal{H}^\varphi|_K$ denotes the restriction of \mathcal{H}^φ to K . In particular, $\dim_{\mathbb{H}} R\mu = \dim_{\mathbb{H}} K$.

Proof. Fix any $k \in \mathbb{N}$. By Lemma 5.1,

$$(5.6) \quad \mathcal{H}^\varphi(R([I]) \cap K) \approx \mathcal{H}^\varphi(R([J]) \cap K) \quad \text{for } I, J \in \mathcal{L}_k(X).$$

Since $0 < \mathcal{H}^\varphi(K) < \infty$ and $\{R([I])\}_{I \in \mathcal{L}_k(X)}$ covers K ,

$$(5.7) \quad 1 \approx \mathcal{H}^\varphi|_K(K) = \sum_{I \in \mathcal{L}_k(X)} \mathcal{H}^\varphi|_K(R([I])).$$

Combining (5.6) and (5.7) gives

$$\mathcal{H}^\varphi|_K(R([I])) \approx \frac{1}{\#\mathcal{L}_k(X)} \quad \text{for } I \in \mathcal{L}_k(X).$$

Then Proposition 2.2 implies that

$$R\mu(R([I])) = \mu(I) \approx \frac{1}{\#\mathcal{L}_k(X)} \approx \mathcal{H}^\varphi|_K(R([I])) \quad \text{for } I \in \mathcal{L}_k(X).$$

Note that $R\mu(R([I])) = 0 = \mathcal{H}^\varphi|_K(R([I]))$ for $I \in \mathcal{A}^k \setminus \mathcal{L}_k(X)$ since μ is supported on X . Hence

$$R\mu(R([I])) \approx \mathcal{H}^\varphi|_K(R([I])) \quad \text{for all } I \in \mathcal{A}^k.$$

This finishes the proof because the collection of sets $\{R([I]) : I \in \bigcup_{k=1}^\infty \mathcal{A}^k\}$ generates the Borel σ -algebra of \mathbb{T}^d . \square

5.2. Proof of Theorem 1.1(iii). The proof of Theorem 1.1(iii) follows immediately by combining the following proposition with (i) and (ii) of Theorem 1.1.

Proposition 5.3. *Let X be a subshift satisfying weak specification and $K = R(X)$. Let μ be the measure of maximal entropy on X . Suppose $s \leq 2$. If $\dim_{\text{H}} R\mu = \dim_{\text{H}} K$, then $\dim_{\text{H}} K = \dim_{\text{B}} K$.*

Proof. By the definition of μ and the assumption on μ ,

$$h(\mu) = h(X) \quad \text{and} \quad \dim_{\text{H}} R\mu = \dim_{\text{H}} K.$$

When $s = 1$. By Proposition 2.4 and Proposition 3.3,

$$\dim_{\text{H}} K = \dim_{\text{H}} R\mu = d_{\text{LY}}(\mu) = \frac{h(\mu)}{\log n_1} = \frac{h(X)}{\log n_1} = \dim_{\text{B}} K.$$

Alternatively, this follows from the result of Furstenberg [20] since T is conformal.

When $s = 2$. Recall $\theta_1 = \alpha_2 \theta_2 = \alpha_2$ from (2.7) and (2.13). By Lemma 2.5,

$$(5.8) \quad \dim_{\text{H}} K = \frac{P}{\log n_2}$$

where

$$(5.9) \quad P = \lim_{k \rightarrow \infty} \frac{1}{k} \log \sum_{J \in \mathcal{L}_k(X_2)} (\#\pi_2^{-1}(J))^{\theta_1}.$$

Let $k \in \mathbb{N}$ and $J \in \mathcal{L}_k(X_2)$. Since $\tau_2 = \pi_2$, Proposition 2.4 implies that

$$(5.10) \quad \pi_2 \mu(J) \approx \frac{(\#\pi_2^{-1}(J))^{\theta_1}}{\exp(kP)}.$$

Since $\mu(I) \approx \exp(-kh(X_1))$ for $I \in \mathcal{L}_k(X_1)$ by [Proposition 2.2](#),

$$(5.11) \quad \pi_2 \mu(J) = \sum_{I \in \pi_2^{-1}(J)} \mu(I) \approx \frac{\#\pi_2^{-1}(J)}{\exp(kh(X_1))}.$$

Combining (5.10) and (5.11) gives

$$(5.12) \quad (\#\pi_2^{-1}(J))^{\theta_1} \approx \exp\left(k \frac{\theta_1}{1-\theta_1} (h(X_1) - P)\right).$$

By [Proposition 2.2](#),

$$(5.13) \quad \#\mathcal{L}_k(X_2) \approx \exp(kh(X_2)).$$

Applying (5.12) and (5.13) to (5.9), we solve that

$$P = \theta_1 h(X_1) + (1 - \theta_1) h(X_2).$$

This shows $\dim_H K = \dim_B K$ by [Proposition 3.3](#) and (5.8). \square

In the above proof, the assumption of $s \leq 2$ allows the potential $\phi^{(2)}$ as in (2.8) to be estimated by combining the Gibbs property of μ and $\tau_2 \mu$. However, this argument does not extend to $s \geq 3$ due to the lack of the information about the regularity of $\tau_i \mu$ for $2 \leq i \leq s-1$. In fact, the examples in [Section 7](#) show that [Proposition 5.3](#) does not hold when $s \geq 3$.

6. BEDFORD-McMULLEN SPONGES

This section is devoted to the results about Bedford-McMullen sponges. The proof of [Theorem 1.2](#) is given in [Section 6.2](#).

6.1. Dimensional results. In this subsection, we include some results of Kenyon and Peres about the dimensions of Bedford-McMullen sponges. Let \mathcal{D} be a nonempty subset of \mathcal{A} . Define $\mathcal{D}_i = \tau_i(\mathcal{D})$ for $1 \leq i \leq s$. Define $Z^{(1)} \equiv 1$ on \mathcal{D}_1 . For $2 \leq i \leq s$, recursively define

$$(6.1) \quad Z^{(i)}(x) = \sum_{y \in \mathcal{D}_{i-1} : \pi_i(y)=x} Z^{(i-1)}(y)^{\alpha_{i-1}} \quad \text{for } x \in \mathcal{D}_i.$$

Define

$$(6.2) \quad Z = \sum_{x \in \mathcal{D}_s} Z^{(s)}(x)^{\alpha_s}.$$

Proposition 6.1 ([26, Theorem 1.2]). *Let $K = R(\mathcal{D}^{\mathbb{N}})$ for some $\emptyset \neq \mathcal{D} \subset \mathcal{A}$. Then*

$$\dim_H K = \frac{\log Z}{\log n_s},$$

where Z is as in (6.2). There is a unique measure of full dimension η_p which is the Bernoulli measure with marginal $p = p(x)_{x \in \mathcal{D}}$, where

$$(6.3) \quad p(x) = \frac{1}{Z} \prod_{i=1}^s Z^{(i)}(\tau_i(x))^{\alpha_{i-1}} \quad \text{for } x \in \mathcal{D}.$$

There is a simple criterion for Bedford-McMullen sponges to have equal Hausdorff and box dimensions. For $1 \leq i \leq s$, define

$$(6.4) \quad f_i(x) = \#\pi_i^{-1}(\tau_i(x)) \quad \text{for } x \in \mathcal{D}.$$

Proposition 6.2 ([26, Proposition 1.3]). *Let $K = R(\mathcal{D}^{\mathbb{N}})$ for some $\emptyset \neq \mathcal{D} \subset \mathcal{A}$. Then $\dim_{\text{H}} K = \dim_{\text{B}} K$ if and only if*

$$(6.5) \quad f_i(x) = f_i(y) \quad \text{for all } 1 \leq i \leq s \text{ and } x, y \in \mathcal{D}.$$

As a direct corollary of [Proposition 6.1](#), the following condition characterizes when the measures of maximal entropy and full dimension coincide.

Corollary 6.3. *Let $K = R(\mathcal{D}^{\mathbb{N}})$ for some $\emptyset \neq \mathcal{D} \subset \mathcal{A}$. Then the measures of maximal entropy and full dimension coincide if and only if*

$$(6.6) \quad \prod_{i=1}^s Z^{(i)}(\tau_i(x))^{\alpha_i-1} = \prod_{i=1}^s Z^{(i)}(\tau_i(y))^{\alpha_i-1} \quad \text{for } x, y \in \mathcal{D}.$$

When $s \leq 3$, (6.6) is equivalent to

$$(6.7) \quad \prod_{i=1}^s f_i(x)^{\theta_{i-1}-1} = \prod_{i=1}^s f_i(y)^{\theta_{i-1}-1} \quad \text{for } x, y \in \mathcal{D}.$$

Proof. Since the measure of maximal entropy on $\mathcal{D}^{\mathbb{N}}$ is the Bernoulli measure with marginal $(1/\#\mathcal{D}, \dots, 1/\#\mathcal{D})$, the first equivalence follows directly from [Proposition 6.1](#). Next we show the second equivalence. Since $f_1(x) = Z^{(1)}(x) = 1$ and $f_2(x) = Z^{(2)}(\tau_2(x))$ for $x \in \mathcal{D}$, (6.7) and (6.6) are equivalent when $s \leq 2$. Suppose $s = 3$. Let $x \in \mathcal{D}$. For $z \in \pi_3^{-1}(\tau_3(x))$, there exists $y \in \mathcal{D}$ such that $\tau_2(y) = z$, and so $\tau_3(y) = \pi_3(\tau_2(y)) = \pi_3(z) = \tau_3(x)$. Then either (6.7) or (6.6) implies $f_2(x) = f_2(y)$, thus $Z^{(2)}(z) = Z^{(2)}(\tau_2(y)) = f_2(y) = f_2(x)$. Hence by (6.1),

$$Z^{(3)}(\tau_3(x)) = \sum_{z \in \pi_3^{-1}(\tau_3(x))} Z^{(2)}(z)^{\alpha_2} = f_3(x) f_2(x)^{\alpha_2}.$$

Thus,

$$\prod_{i=1}^3 Z^{(i)}(\tau_i(x))^{\alpha_i-1} = f_2(x)^{\alpha_2-1} (f_3(x) f_2(x)^{\alpha_2})^{\alpha_3-1} = f_2(x)^{\theta_1-1} f_3(x)^{\theta_2-1},$$

which finishes the proof. \square

It is easy to see that (6.5) is equivalent to (6.7) if $s \leq 2$, but stronger than (6.7) if $s = 3$. This motivates us to give the examples in [Section 7](#).

6.2. Proof of Theorem 1.2. Let X be a subshift and μ be a measure on X . For a gauge function φ , define

$$\Theta_k^\varphi(\mu, x) = \frac{R\mu(Q_k(x))}{\varphi(n_s^{-k})} \quad \text{for } x \in X \text{ and } k \in \mathbb{N}.$$

Write Θ_k^φ as Θ_k^γ when $\varphi(r) = r^\gamma$ for some $\gamma \geq 0$. Let $E[X]$ and $\text{Var}(X)$ respectively denote the expectation and variance of a random variable X .

Proof of Theorem 1.2. By Theorem 1.1(i), it remains to show that if $0 < \mathcal{H}^\varphi(K) < \infty$ for some gauge function φ , then $\dim_{\text{H}} K = \dim_{\text{B}} K$.

Suppose $0 < \mathcal{H}^\varphi(K) < \infty$. Write $K = R(\mathcal{D}^{\mathbb{N}})$ for some $\mathcal{D} \subset \mathcal{A}$. Let μ be the measure of maximal entropy on $\mathcal{D}^{\mathbb{N}}$, that is, μ is the Bernoulli measure with marginal $(1/\#\mathcal{D}, \dots, 1/\#\mathcal{D})$. Let $x = (x_j)_{j=1}^\infty \in \mathcal{D}^{\mathbb{N}}$ be a random sequence with law μ . For $k \in \mathbb{N}$ and $1 \leq j \leq k$, define the random variables

$$(6.8) \quad X_j^{(k)} = \log \# \tau_i^{-1}(\tau_i(x_j)) \quad \text{for } 1 \leq i \leq s \text{ and } \lfloor \theta_{i-1}k \rfloor + 1 \leq j \leq \lfloor \theta_i k \rfloor.$$

Note that x_1, \dots, x_k are i.i.d. random variables with common law $\sum_{y \in \mathcal{D}} 1/(\#\mathcal{D}) \delta_y$. Then $X_1^{(k)}, \dots, X_k^{(k)}$ are independent and uniformly bounded. Write $E_k = \sum_{j=1}^k E[X_j^{(k)}]$ and $V_k^2 = \sum_{j=1}^k \text{Var}(X_j^{(k)})$. For $k \in \mathbb{N}$, define

$$(6.9) \quad L(k) = \log((\#\mathcal{D})^k \varphi(n_s^{-k})).$$

For $x = (x_j)_{j=1}^\infty \in \mathcal{D}^{\mathbb{N}}$ and $k \in \mathbb{N}$, (2.17) implies

$$(6.10) \quad \begin{aligned} \#\Gamma_k(x) &= \#\left\{ I \in \mathcal{L}_k(\mathcal{D}^{\mathbb{N}}) : \tau_i(I|_{\lfloor \theta_{i-1}k \rfloor}) = \tau_i(x|_{\lfloor \theta_{i-1}k \rfloor}) \text{ for } 1 \leq i \leq s \right\} \\ &= \prod_{i=1}^s \prod_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} \# \tau_i^{-1}(\tau_i(x_j)). \end{aligned}$$

Since $\mu(I) = 1/(\#\mathcal{D})^k$ for $I \in \mathcal{D}^k$, Lemma 2.6 shows

$$(6.11) \quad R\mu(Q_k(x)) = \sum_{I \in \Gamma_k(x)} \mu(I) = \frac{\#\Gamma_k(x)}{(\#\mathcal{D})^k}.$$

Combining (6.8), (6.9), (6.10) and (6.11) gives

$$(6.12) \quad \Theta_k^\varphi(\mu, x) = \exp\left(\sum_{j=1}^k X_j^{(k)} - L(k)\right) \quad \text{for } x \in \mathcal{D}^{\mathbb{N}}.$$

By Proposition 5.2 and the upper density theorem for the Hausdorff measures (see Proposition B.1 or [10]), there exists $C > 0$ such that

$$\mu \left\{ x : \frac{1}{C} \leq \limsup_{k \rightarrow \infty} \Theta_k^\varphi(\mu, x) \leq C \right\} = 1.$$

From this and (6.12), there exists $M > 0$ such that

$$(6.13) \quad \mathbb{P} \left\{ \limsup_{k \rightarrow \infty} \left| \sum_{j=1}^k X_j^{(k)} - L(k) \right| < M \right\} = 1,$$

where \mathbb{P} denotes the underlying probability (that is, a probabilistic notation for μ).

Suppose on the contrary that $\dim_{\text{H}} K < \dim_{\text{B}} K$. It follows from Proposition 6.2 that $\#\tau_{i_0}^{-1}(\tau_{i_0}(x)) \neq \#\tau_{i_0}^{-1}(\tau_{i_0}(y))$ for some $1 \leq i_0 \leq s$ and $x, y \in \mathcal{D}$. Let Y be the random

variable with law $\sum_{x \in \mathcal{D}} (1/\#\mathcal{D}) \delta_{\# \tau_{i_0}^{-1}(\tau_{i_0}(x))}$, then $\text{Var}(Y) > 0$. By (6.8), Y has the same law as $X_j^{(k)}$ for $\lfloor \theta_{i-1}k \rfloor + 1 \leq j \leq \lfloor \theta_i k \rfloor$. Then by the independence of $X_j^{(k)}$,

$$V_k^2 \geq \sum_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} \text{Var}(X_j^{(k)}) \gtrsim \text{Var}(Y)(\theta_i - \theta_{i-1})k \rightarrow \infty,$$

as $k \rightarrow \infty$. By Lyapunov's Central Limit Theorem (see e.g. [7, Theorem 7.1.2]), the normalized random sum $Y_k = (\sum_{j=1}^k X_j^{(k)} - E_k)/V_k$ converges to the standard normal random variable $N(0, 1)$ in distribution. Since the distribution of $N(0, 1)$ is absolutely continuous and $\lim_{k \rightarrow \infty} M/V_k = 0$,

$$(6.14) \quad \mathbb{P} \left\{ \left| \sum_{j=1}^k X_j^{(k)} - L(k) \right| < M \right\} = \mathbb{P} \left\{ \left| Y_k - \frac{L(k) - E_k}{V_k} \right| < \frac{M}{V_k} \right\} \rightarrow 0$$

as $k \rightarrow \infty$. This implies

$$\mathbb{P} \left\{ \limsup_{k \rightarrow \infty} \left| \sum_{j=1}^k X_j^{(k)} - L(k) \right| < M \right\} \leq \limsup_{k \rightarrow \infty} \mathbb{P} \left\{ \left| \sum_{j=1}^k X_j^{(k)} - L(k) \right| < M \right\} = 0,$$

which contradicts (6.13). \square

Remark 6.4. The proof of Theorem 1.2 works for some other subshift X satisfying weak specification if we can write $\log R\mu(Q_k(x))$ as a sequence of random variables such that certain limit theorem can be applied to conclude an analog of (6.14). Next we give an example to illustrate it. Let m_1, m_2, m_3 be integers such that $m_1 > m_2 > m_3 \geq 2$, $m_1, m_2 \geq 3$. Let $\mathcal{A}, \theta_i, \pi_i, \tau_i, 1 \leq i \leq s = 3$ be defined as in Section 2.1. Let

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Enumerate \mathcal{D} as $\{z_1, \dots, z_6\}$ according to the above order. Let $A = (A_{i,j})$ be a 6×6 matrix such that $A_{i,i} = 0$ and $A_{i,j} = 1$ if $i \neq j$ for $1 \leq i, j \leq 6$. Consider the following subshift of finite type

$$\Sigma_A = \{(x_k)_{k=1}^\infty \in \mathcal{D}^\mathbb{N} : \forall k \in \mathbb{N}, \exists 1 \leq i, j \leq 6, x_k = z_i, x_{k+1} = z_j, A_{i,j} = 1\}.$$

Let $K = R(\Sigma_A)$. By [35] (see also [41]), the unique measure of maximal entropy μ on Σ_A is the Markov measure generated by the probability vector $(1/6, \dots, 1/6)$ and the random matrix $A/5$. By computation, $\tau_2\mu$ is the Markov measure on $\tau_2(\Sigma_A)$ generated by

$$(1/6, 1/3, 1/2) \text{ and } \begin{pmatrix} 0 & 2/5 & 3/5 \\ 1/5 & 1/5 & 3/5 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}. \text{ Since the computation shows that } \tau_2\mu \text{ is not the}$$

measure of maximal entropy on $\tau_2(\Sigma_A)$, it follows from Theorem 3.1(b) that $\dim_H K < \dim_B K$. By the definition of \mathcal{D} , a similar estimate as in the proof of Theorem 1.2 gives

$$\log R\mu(Q_k(x)) \approx \theta_1 k + 3(1 - \theta_2)k + \sum_{j=\lfloor \theta_1 k \rfloor + 1}^{\lfloor \theta_2 k \rfloor} Y((\tau_2 x)_j),$$

where $Y: \tau_2 \mathcal{D} \rightarrow \{1, 2, 3\}$ is a function defined as $Y((0, 0)^t) = 1$, $Y((1, 0)^t) = 2$ and $Y((2, 1)^t) = 3$. Since the law of $\tau_2 x$ is the Markov measure $\tau_2 \mu$ if x is a random sequence with law μ , we can apply the central limit theorem for Markov chains (see e.g. [11, Example 8.3.4]) and adapt the proof of Theorem 1.2 to conclude that $\mathcal{H}^\varphi(K) = 0$ or ∞ for every gauge function φ .

6.3. Infinite Hausdorff measure. In this subsection, we provide a sufficient condition (see Proposition 6.6) for Bedford-McMullen sponges to have infinite Hausdorff measures at their respective Hausdorff dimensions. This is based on the method of Peres [37] and a relation (see Lemma 6.8) satisfied by the measure of full dimension. In [37, Section 5], Peres mentioned that [37, Theorem 1 and Proposition 2] can extend to Bedford-McMullen sponges. However, he did not give a detailed justification. In fact, the proof of [37, Proposition 2] does not work for Bedford-McMullen sponges, as indicated by the examples in Section 7. For the extension of [37, Theorem 1], it is not clear how to construct a measure with zero density like that in the proof of [37, Theorem 1] since the measure of full dimension becomes more complicated when $s \geq 3$ (see Proposition 6.1).

Throughout this subsection we fix a nonempty subset \mathcal{D} of \mathcal{A} and let $K = R(\mathcal{D}^\mathbb{N})$. Let $p = (p(x))_{x \in \mathcal{D}}$ be the probability vector given in (6.3). Write $\gamma = \dim_H K$. For a finite set \mathcal{B} and probability vectors $q = (q(x))_{x \in \mathcal{B}}$ and $q' = (q'(x))_{x \in \mathcal{B}}$, following [37] we define

$$(6.15) \quad \Delta(q \| q') = \sum_{x \in \mathcal{B}} (q(x) - q'(x)) \log q(x),$$

where we require $q'(x) = 0$ if $q(x) = 0$.

The following proposition is essentially contained in [37]. For completeness, we include a proof of Proposition 6.5 in Appendix A.

Proposition 6.5 (Peres [37]). *If there exist a probability vector q on \mathcal{D} and $c > 0$ such that*

$$(6.16) \quad \sum_{i=1}^s \Delta(\tau_i p \| \tau_i q) \int_{\theta_{i-1} k}^{\theta_i k} \frac{1}{\log t} dt \geq c \frac{k}{(\log k)^2} \quad \text{for large } k \in \mathbb{N},$$

then $\mathcal{H}^\varphi(K) = \infty$, where

$$(6.17) \quad \varphi(r) = r^\gamma \exp \left(-\tilde{c} \frac{|\log r|}{(\log |\log r|)^2} \right), \quad r > 0,$$

for some $\tilde{c} > 0$ depending on c explicitly. In particular, $\mathcal{H}^\gamma(K) = \infty$.

In the remaining part of this subsection, we prove the following proposition which is a sufficient condition such that the Hausdorff measures of Bedford-McMullen sponges are infinite at their Hausdorff dimensions. We do not expect condition (6.18) to be necessary, but it is sufficient for determining the Hausdorff measure of the examples in Section 7 at their respective Hausdorff dimensions. For a probability vector $q = (q(x))_{x \in \mathcal{B}}$ on a finite set \mathcal{B} , we say q is *uniform* if $q = (1/\#\mathcal{B}, \dots, 1/\#\mathcal{B})$.

Proposition 6.6. *If*

$$(6.18) \quad 1 \leq \#\{1 \leq i \leq s: \tau_i p \text{ is not uniform}\} \leq 2,$$

then $\mathcal{H}^{\dim_{\mathbb{H}} K}(K) = \infty$.

Proposition 6.6 extends some results in [37] since for Bedford-McMullen carpets, the upper bound in (6.18) trivially hold when $d = 2$ and the lower bound in (6.18) follows from $\dim_{\mathbb{H}} K < \dim_{\mathbb{B}} K$ and **Theorem 3.1(b)**. Next we give two lemmas for the proof of **Proposition 6.6**.

Lemma 6.7. *If q is uniform, then $\Delta(q\|q') = 0$; if q' is uniform but q is not uniform, then $\Delta(q\|q') > 0$.*

Proof. The first statement is by a direct computation. The second statement follows from

$$\Delta(q\|q') = \frac{1}{2\#\mathcal{B}} \sum_{x \in \mathcal{B}} \sum_{y \in \mathcal{B}} (q(x) - q'(y))(\log q(x) - \log q'(y)) > 0,$$

when q' is uniform. □

The following lemma is the key ingredient for the verification of (6.16) when $s \geq 3$.

Lemma 6.8. *For every probability vector q on \mathcal{D} ,*

$$(6.19) \quad \sum_{i=1}^s (\theta_i - \theta_{i-1}) \Delta(\tau_i p \|\tau_i q) = 0.$$

Proof. For $1 \leq i \leq s$, write $p_i = \tau_i p$, $q_i = \tau_i q$, and $\Delta_i = \Delta(p_i\|q_i)$. By (6.15),

$$\begin{aligned} \Delta_i &= \sum_{x \in \mathcal{D}_i} (p_i(x) - q_i(x)) \log p_i(x) \\ &= \sum_{x \in \mathcal{D}_i} \log p_i(x) \sum_{y \in \tau_i^{-1}(x)} (p_1(y) - q_1(y)) \quad (\text{by } p_i = \tau_i p_1, q_i = \tau_i q_1) \\ &= \sum_{x \in \mathcal{D}_i} \sum_{y \in \tau_i^{-1}(x)} (p_1(y) - q_1(y)) \log p_i(\tau_i(y)) \quad (\text{by } \tau_i(y) = x \text{ for } y \in \tau_i^{-1}(x)) \\ &= \sum_{y \in \mathcal{D}_1} (p_1(y) - q_1(y)) \log p_i(\tau_i(y)). \end{aligned}$$

Then

$$\sum_{i=1}^s (\theta_i - \theta_{i-1}) \Delta_i = \sum_{x \in \mathcal{D}_1} (p_1(x) - q_1(x)) \sum_{i=1}^s (\theta_i - \theta_{i-1}) \log p_i(\tau_i(x)).$$

Since $\sum_{x \in \mathcal{D}_1} p_1(x) = \sum_{x \in \mathcal{D}_1} q_1(x) = 1$, it suffices to show

$$(6.20) \quad \sum_{i=1}^s (\theta_i - \theta_{i-1}) \log p_i(\tau_i(x)) = -\log Z \quad \text{for } x \in \mathcal{D}_1.$$

By (6.3), for $x \in \mathcal{D}_1$ and $2 \leq i \leq s$,

$$p_i(x) = p_i(\tau_i(x)) \frac{1}{Z^{(i)}(\tau_i(x))} \prod_{j=1}^{i-1} Z^{(j)}(\tau_j(x))^{\alpha_j - 1}.$$

Thus

$$\begin{aligned}
& \sum_{i=1}^s (\theta_i - \theta_{i-1}) \log p_i(\tau_i(x)) \\
&= \log p_1(x) - \sum_{i=2}^s (\theta_{i-1} - \theta_i) Z^{(i)}(\tau_i(x)) - \sum_{i=2}^s \sum_{j=1}^{i-1} (\theta_i - \theta_{i-1})(\alpha_j - 1) \log Z^{(j)}(\tau_j(x)) \\
&= \log p_1(x) - \sum_{i=2}^s (\theta_{i-1} - \theta_i) Z^{(i)}(\tau_i(x)) - \sum_{j=1}^{s-1} \sum_{i=j+1}^s (\theta_i - \theta_{i-1})(\alpha_j - 1) \log Z^{(j)}(\tau_j(x)) \\
&= \log p_1(x) - \sum_{i=2}^s (\theta_{i-1} - \theta_i) Z^{(i)}(\tau_i(x)) - \sum_{j=1}^{s-1} (1 - \theta_j)(\alpha_j - 1) \log Z^{(j)}(\tau_j(x)) \\
&= \log p_1(x) - \sum_{i=1}^s (\alpha_i - 1) Z^{(i)}(\tau_i(x)) \\
&= -\log Z,
\end{aligned}$$

where the second equality is by changing the index of i, j ; the second last equality is by $Z^{(1)} \equiv 1$ and $\theta_i \alpha_i = \theta_{i-1}$ for $1 \leq i \leq s$; the last equality is by (6.3). \square

Now we are ready to prove [Proposition 6.6](#).

Proof of Proposition 6.6. For a probability vector q on \mathcal{D} and $1 \leq i \leq s$, write $p_i = \tau_i p$, $q_i = \tau_i q$, and $\Delta_i = \Delta(p_i \| q_i)$. Define $I = \{1 \leq i \leq s : p_i \text{ is not uniform}\}$. It follows from [Lemma 6.7](#) that $\Delta_i = 0$ for $i \notin I$. By [Lemma 6.8](#),

$$(6.21) \quad \sum_{i \in I} (\theta_i - \theta_{i-1}) \Delta_i = \sum_{i=1}^s (\theta_i - \theta_{i-1}) \Delta_i = 0.$$

We first show $\#I \neq 1$. Suppose otherwise that $I = \{i_1\}$. Take a probability vector q on \mathcal{D} such that $\tau_{i_1} q$ is uniform. Then $\Delta_{i_1} > 0$ by [Lemma 6.7](#), which contradicts (6.21). Then $\#I = 2$ by (6.18).

Write $I = \{i_1, i_2\}$ for some $1 \leq i_1 < i_2 \leq s$. Take a probability vector q on \mathcal{D} such that $\tau_{i_1} q$ is uniform. Then $\Delta_{i_1} > 0$ by [Lemma 6.7](#). By (6.21),

$$\Delta_{i_2} = -\frac{\theta_{i_1} - \theta_{i_1-1}}{\theta_{i_2} - \theta_{i_2-1}} \Delta_{i_1} < 0.$$

Since $\Delta_i = 0$ for $i \neq I$,

$$\begin{aligned}
(6.22) \quad & \sum_{i=1}^s \Delta_i \int_{\theta_{i-1}k}^{\theta_i k} \frac{1}{\log t} dt \\
&= \Delta_{i_1} (\theta_{i_1} - \theta_{i_1-1}) \left(\frac{1}{\theta_{i_1} - \theta_{i_1-1}} \int_{\theta_{i_1-1}k}^{\theta_{i_1}k} \frac{1}{\log t} dt - \frac{1}{\theta_{i_2} - \theta_{i_2-1}} \int_{\theta_{i_2-1}k}^{\theta_{i_2}k} \frac{1}{\log t} dt \right).
\end{aligned}$$

By a change of variable, for $0 < a < b \leq c < d$ there exists $\delta > 0$ such that

$$(6.23) \quad \frac{1}{b-a} \int_{ak}^{bk} \frac{1}{\log t} dt - \frac{1}{d-c} \int_{ck}^{dk} \frac{1}{\log t} dt \geq \delta \frac{k}{(\log k)^2} \quad \text{for } k \in \mathbb{N} \cap \left[\frac{2}{a}, \infty \right).$$

Combining (6.22) and (6.23) verifies (6.16). Finally, the proof is completed by Proposition 6.5. \square

Remark 6.9. From the proof of Proposition 6.6 we see that

$$(6.24) \quad \#\{1 \leq i \leq s: \tau_i p \text{ is not uniform}\} \neq 1.$$

Based on (6.24) there is an alternative way to prove that for a Bedford-McMullen carpet K , $0 < \mathcal{H}^\varphi(K) < \infty$ implies $\dim_{\text{H}} K = \dim_{\text{B}} K$. Suppose otherwise $\dim_{\text{H}} K < \dim_{\text{B}} K$. Combining Proposition 5.2 and Theorem 3.1(b) gives $\#\{1 \leq i \leq s: \tau_i p \text{ is not uniform}\} = 1$ which contradicts (6.24).

7. EXAMPLES

In this section, we present several Bedford-McMullen sponges $K = R(\mathcal{D}^{\mathbb{N}})$, $\emptyset \neq \mathcal{D} \subset \mathcal{A}$ such that $\dim_{\text{H}} R\mu = \dim_{\text{H}} K < \dim_{\text{B}} K$, where μ is the measure of maximal entropy on $\mathcal{D}^{\mathbb{N}}$. By Theorem 1.2, no gauge function exists for such sponges. For a finite set \mathcal{B} and a probability vector $p = (p(x))_{x \in \mathcal{B}}$, let η_p denote the Bernoulli measure on $\mathcal{B}^{\mathbb{N}}$ with marginal p .

Example 7.1. Let

$$\Lambda = \text{diag}(64, 16, 8)$$

and

$$\mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Define $K = R(\mathcal{D}^{\mathbb{N}})$. By Proposition 3.3 and Proposition 6.1,

$$\dim_{\text{B}} K = \frac{\log 360}{12 \log 2} \quad \text{and} \quad \dim_{\text{H}} K = \frac{\log 18}{6 \log 2}.$$

The measure of maximal entropy is η_p with $p = p_1 = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$. Note that $\tau_2 \eta_{p_1} = \eta_{p_2}$ and $\tau_3 \eta_{p_1} = \eta_{p_3}$, where $p_2 = (1/6, 1/6, 1/6, 1/6, 1/3)$ and $p_3 = (2/3, 1/3)$. By Proposition 2.3,

$$\dim_{\text{H}} R\eta_p = d_{\text{LY}}(\eta_p) = \frac{\log 18}{6 \log 2}.$$

Hence

$$\dim_{\text{H}} R\eta_p = \dim_{\text{H}} K < \dim_{\text{B}} K.$$

Since $\#\{1 \leq i \leq 3: p_i \text{ is not uniform}\} = 2$, it follows from Proposition 6.6 that

$$\mathcal{H}^{\dim_{\text{H}} K}(K) = \infty.$$

Next we give similar examples when $d \geq 4$.

Example 7.2. For $d \geq 4$, let

$$\Lambda = \text{diag}(2^{2^{d-1}}, 2^{2^{d-2}}, \dots, 4, 2)$$

and

$$\mathcal{D} = \left\{ \begin{pmatrix} \{0, 1, 2, 3\} \\ \{0\} \\ \{0, 1\} \\ \{0\} \\ \{0, 1\} \\ \vdots \\ \{0, 1\} \end{pmatrix}, \begin{pmatrix} \{0\} \\ \{0, 1, 2, 3, 4, 5, 6, 7\} \\ \{0\} \\ \{1\} \\ \{0, 1\} \\ \vdots \\ \{0, 1\} \end{pmatrix} \right\}.$$

Write the left and right collections of digits above respectively as \mathcal{D}_1 and \mathcal{D}_2 . Then $\mathcal{D} = \mathcal{D}_1 \sqcup \mathcal{D}_2$. Since for $x \in \mathcal{D}_1$ and $y \in \mathcal{D}_2$,

$$\#\pi_2^{-1}(\tau_2(x)) = 4 \neq 1 = \#\pi_2^{-1}(\tau_2(y)),$$

[Proposition 6.2](#) implies $\dim_{\mathbb{H}} K < \dim_{\mathbb{B}} K$. Note that $\alpha_i = 1/2$ for $2 \leq i \leq d$. Then for $x \in \mathcal{D}_1$ and $y \in \mathcal{D}_2$, by [\(6.1\)](#),

$$\begin{aligned} Z^{(2)}(\tau_2(x)) &= 4 & \text{and} & & Z^{(2)}(\tau_2(y)) &= 1 \\ (7.1) \quad Z^{(3)}(\tau_3(x)) &= 2 & \text{and} & & Z^{(3)}(\tau_3(y)) &= 8 \\ Z^{(4)}(\tau_4(x)) &= 2\sqrt{2} & \text{and} & & Z^{(4)}(\tau_4(y)) &= 2\sqrt{2}. \end{aligned}$$

If $i > 4$ and $x, y \in \mathcal{D}$, since $\#\pi_i^{-1}(\tau_i(x)) = \#\pi_i^{-1}(\tau_i(y))$,

$$(7.2) \quad Z^{(i)}(\tau_i(x)) = Z^{(i)}(\tau_i(y)).$$

By computation,

$$\prod_{i=1}^3 Z^{(i)}(\tau_i(x))^{\alpha_i-1} = 2^{-3/2} \quad \text{for } x \in \mathcal{D}.$$

Then [Corollary 6.3](#) shows that $\dim_{\mathbb{H}} R\eta_p = \dim_{\mathbb{H}} K$, where $p = (1/\#\mathcal{D}, \dots, 1/\#\mathcal{D})$. Hence $\dim_{\mathbb{H}} R\eta_p = \dim_{\mathbb{H}} K < \dim_{\mathbb{B}} K$. Since [\(6.3\)](#) implies that p_1 and $\tau_i p, i \geq 4$ are uniform while $\tau_2 p$ and $\tau_3 p$ are not uniform, it follows from [Proposition 6.6](#) that $\mathcal{H}^{\dim_{\mathbb{H}} K}(K) = \infty$.

Finally, we conclude with some remarks on open problems. Let X be a subshift satisfying weak specification and $K = R(X)$. When $s = 2$ and $\dim_{\mathbb{H}} K < \dim_{\mathbb{B}} K$, [Theorem 1.1](#) implies that $\mathcal{H}^{\dim_{\mathbb{H}} K}(K)$ is either zero or infinite; however, whether $\mathcal{H}^{\dim_{\mathbb{H}} K}(K) = \infty$ holds remains open. As illustrated by [Examples 7.1](#) and [7.2](#), the current method cannot show the equivalence between [\(A\)](#) and [\(B\)](#) when $s \geq 3$. We expect that the proof of [Theorem 1.1](#) may extend to certain general linear automorphisms on \mathbb{T}^2 , but handling non-homogeneous IFSs, such as those in Gatzouras–Lalley and Barański settings, appears more challenging. It is worth noting that Qiu and Wang [[38](#), Corollary 1.2] recently showed that given a Barański carpet K , $\dim_{\mathbb{H}} K = \dim_{\mathbb{B}} K$ is strictly stronger than $0 < \mathcal{H}^{\dim_{\mathbb{H}} K}(K) < \infty$.

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APPENDIX A. PROOF OF PROPOSITION 6.5

Proof of Proposition 6.5. The proof is essentially contained in [37]. For completeness we give a sketch of the proof. The strategy is to construct a measure ν on $\mathcal{D}^{\mathbb{N}}$ such that $\limsup_{k \rightarrow \infty} R\nu(Q_k(x))/\varphi(n_s^{-k}) = 0$ for ν -a.e. x .

For $1 \leq i \leq s$, write $p_i = \tau_i p$, $q_i = \tau_i q$, and $\Delta_i = \Delta(p_i \| q_i)$. For $\delta > 0$ and $j \geq 2$, following [37] we define

$$\xi^{(j)} = \left(1 - \frac{\delta}{\log j}\right) p + \frac{\delta}{\log j} q.$$

and $\xi^{(1)} = p$. Define the product measure on $\mathcal{D}^{\mathbb{N}}$ by

$$\nu = \prod_{j=1}^{\infty} \xi^{(j)}.$$

For $1 \leq i \leq s$ and $1 \leq j \leq k$, $k \in \mathbb{N}$, let $X_{i,j}^{(k)}$ be the random variable on \mathcal{D}_j with law $\sum_{y \in \mathcal{D}_i} \tau_i \xi^{(j)}(y) \delta_{\log \tau_i \xi^{(j)}(y)}$. Then $X_{i,j}^{(k)}$ is well defined since for $y \in \mathcal{D}_i$ there is $x \in \mathcal{D}$ with $\tau_i(x) = y$ such that $\tau_i \xi^{(j)}(y) \geq (1 - \frac{\delta}{\log j}) p(x) > 0$ by (6.3). By (2.17) and Lemma 2.6, for $k \in \mathbb{N}$ and $x = (x_j) \in \mathcal{D}^{\mathbb{N}}$,

$$(A.1) \quad \log R\nu(Q_k(x)) = \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1} k \rfloor + 1}^{\lfloor \theta_i k \rfloor} \log \tau_i \xi^{(j)}(\tau_i(x_j)) = \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1} k \rfloor + 1}^{\lfloor \theta_i k \rfloor} X_{i,j}^{(k)}.$$

Then $\log R\nu(Q_k(x))$ is a sum of independent variables. By (6.16) and Lemma 6.7, there is $1 \leq i_0 \leq s$ such that $p_{i_0} := \tau_{i_0} p$ is not uniform. Let Y_{i_0} be the random variable on \mathcal{D}_{i_0} with the law $\sum_{y \in \mathcal{D}_{i_0}} p_{i_0}(y) \delta_{\log p_{i_0}(y)}$. Then $\text{Var}(X_{i_0,j}^{(k)}) \geq \text{Var}(Y_{i_0})/2 > 0$ for j large enough since $\tau_{i_0} \xi^{(j)} = \left(1 - \frac{\delta}{\log j}\right) p_{i_0} + \frac{\delta}{\log j} q_{i_0}$. Then for k large,

$$\text{Var}(\log R\nu(Q_k(x))) \geq \sum_{j=(\lfloor \theta_{i_0} k \rfloor + \lfloor \theta_{i_0-1} k \rfloor)/2}^{\lfloor \theta_{i_0} k \rfloor} \text{Var}(X_{i_0,j}^{(k)}) \geq \frac{\text{Var}(Y_{i_0})(\theta_{i_0} - \theta_{i_0-1})}{4} k \gtrsim k.$$

Thus the law of the iterated logarithm (see, e.g. [7, Theorem 7.5.1]) implies that

$$(A.2) \quad \log R\nu(Q_k(x)) \leq E_k(\nu) + O\left(\sqrt{k \log \log k}\right) \quad \text{for } \nu\text{-a.e. } x.$$

where $E_k(\nu) := \int \log R\nu(Q_k(x)) d\nu(x)$. By Taylor's theorem, for probability vectors q, q' and $\varepsilon > 0$,

$$(A.3) \quad H((1 - \varepsilon)q + \varepsilon q') = H(q) + \varepsilon \Delta(q \| q') + O(\varepsilon^2),$$

where $H(\cdot)$ is the *Shannon entropy*, that is, for a probability vector $q = (q(x))_{x \in \mathcal{B}}$,

$$H(q) = \sum_{x \in \mathcal{B}} -q(x) \log q(x).$$

Then

$$\begin{aligned}
E_k(\nu) &= \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} \sum_{x \in \mathcal{D}_i} \tau_i \xi^{(j)}(x) \log \tau_i \xi^{(j)}(x) \\
&= \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} -H(\tau_i \xi^{(j)}) \\
&= \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} -H\left(\left(1 - \frac{\delta}{\log j}\right) p_i + \frac{\delta}{\log j} q_i\right) \\
&= \sum_{i=1}^s \sum_{j=\lfloor \theta_{i-1}k \rfloor + 1}^{\lfloor \theta_i k \rfloor} -H(p_i) - \delta \Delta_i \frac{1}{\log j} + O\left(\frac{\delta^2}{(\log j)^2}\right) \\
&= -k \log Z - \delta \sum_{i=1}^s \Delta_i \sum_{j=\lfloor \theta_i k \rfloor + 1}^{\lfloor \theta_i k \rfloor} \frac{1}{\log j} + O(1) + \delta^2 O(1),
\end{aligned}$$

where the last equality is by [Proposition 6.1](#) and $\sum_{j \geq 2} 1/(\log j)^2 < \infty$. Since

$$\sum_{j=a}^b \frac{1}{\log j} = \int_a^b \frac{1}{\log t} dt + O(1) \quad \text{for } 2 \leq a < b,$$

we have

$$(A.4) \quad E_k(\nu) = -k \log Z - \delta \sum_{i=1}^s \Delta_i \int_{\theta_{i-1}k}^{\theta_i k} \frac{1}{\log t} dt + O(1).$$

Let $\tilde{c} > 0$ be small such that $\tilde{c}M < c\delta/2$, where $M > 0$ is large such that for $k \in \mathbb{N} \cap [2, \infty)$, $|\log n_s^{-k}|/(\log |\log n_s^{-k}|)^2 \leq Mk/(\log k)^2$. Take φ as in [\(6.17\)](#). Combining [\(A.4\)](#), [\(6.16\)](#), [\(6.17\)](#) and [\(A.2\)](#) gives

$$\limsup_{k \rightarrow \infty} \log R\nu(Q_k(x)) - \log \varphi(n_s^{-k}) = -\infty \quad \text{for } \nu\text{-a.e. } x.$$

By [Lemma 2.9](#), this shows $\mathcal{H}^\varphi(K) = \infty$. □

APPENDIX B. AN UPPER DENSITY THEOREM FOR HAUSDORFF MEASURES

This section is devoted to the proof of the following density theorem for Hausdorff measures with respect to gauge functions.

Proposition B.1. *There exists $C > 0$ such that if $E \subset \mathbb{T}^d$ satisfies $0 < \mathcal{H}^\varphi(E) < \infty$ for some gauge function φ , then for \mathcal{H}^φ -a.e. $x \in E$,*

$$\frac{1}{C} \leq \limsup_{k \rightarrow \infty} \frac{\mathcal{H}^\varphi(E \cap Q_k(x))}{\varphi(n_s^{-k})} \leq C,$$

where $Q_k(x)$ are approximate cubes defined in [\(2.16\)](#).

We first introduce a measure of Hausdorff type defined using approximate cubes. Given a gauge function φ , for $E \subset \mathbb{T}^d$ and $k \in \mathbb{N}$, define

$$\tilde{\mathcal{H}}_k^\varphi(E) = \inf \left\{ \sum_i \varphi(n_s^{-k_i}) : \{Q_i\} \text{ is a cover of } E \text{ such that } Q_i \in \mathcal{Q}_{k_i}, k_i \geq k \right\},$$

where \mathcal{Q}_{k_i} denotes the collection of k_i -level approximate cubes as in (2.19), and

$$\tilde{\mathcal{H}}^\varphi(E) = \lim_{k \rightarrow \infty} \tilde{\mathcal{H}}_k^\varphi(E).$$

Proposition B.1 follows from a combination of **Lemma B.2** and **Proposition B.3**.

Lemma B.2. *There exists $C > 0$ such that for each gauge function φ and $E \subset \mathbb{T}^d$,*

$$\frac{\tilde{\mathcal{H}}^\varphi(E)}{C} \leq \mathcal{H}^\varphi(E) \leq C\tilde{\mathcal{H}}^\varphi(E).$$

Proof. By (2.16) there exists $c \geq 1$ such that $n_s^{-k}/c \leq |Q| \leq cn_s^{-k}$ for each $Q \in \mathcal{Q}_k$. Fix $N \in \mathbb{N}$ such that $cn_s^{-N} \leq 1$.

We first show $\mathcal{H}^\varphi(E) \leq C_1 \tilde{\mathcal{H}}^\varphi(E)$ for some $C_1 \geq 1$. Let $k \in \mathbb{N}$ and $\{Q_i\}_{i=1}^\infty$ be a cover of E with $Q_i \in \mathcal{Q}_{k_i}$, $k_i \geq k$. Since $|Q| \approx |Q_i|$ for each $Q \in \mathcal{Q}_{k_i+N}$, there exists $C_1 \geq 1$ such that each Q_i can be covered by $Q_{i,1}, \dots, Q_{i,C_1} \in \mathcal{Q}_{k_i+N}$. Then $\{Q_{i,j}\}$ is cover of E and $|Q_{i,j}| \leq cn_s^{-k_i-N} \leq n_s^{-k_i}$. Hence

$$\mathcal{H}_{n_s^{-k}}^\varphi(E) \leq \sum_i \sum_j \varphi(|Q_{i,j}|) \leq \sum_i \sum_j \varphi(n_s^{-k_i}) \leq C_1 \sum_i \varphi(n_s^{-k_i}).$$

Taking infimum over covers $\{Q_i\}$ and letting $k \rightarrow \infty$ give $\mathcal{H}^\varphi(E) \leq C_1 \tilde{\mathcal{H}}^\varphi(E)$.

Next we show $\tilde{\mathcal{H}}^\varphi(E) \leq C_2 \mathcal{H}^\varphi(E)$ for some $C_2 \geq 1$. Let $\delta > 0$ and $\{U_i\}$ be a cover of E with $|U_i| \leq \delta$. Write $k_i = \lfloor -\log|U_i|/\log n_s \rfloor + 1$ and $k(\delta) = \lfloor -\log \delta / \log n_s \rfloor + 1$. Since $|Q| \approx |U_i|$ for each $Q \in \mathcal{Q}_{k_i}$, there exists $C_2 \geq 1$ such that each U_i can be covered by $Q_{i,1}, \dots, Q_{i,C_2} \in \mathcal{Q}_{k_i}$. Then $\{Q_{i,j}\}$ is a cover of E by approximate cubes $Q_{i,j} \in \mathcal{Q}_{k_i}$ and $n_s^{-k_i} \leq |U_i|$. Hence

$$\tilde{\mathcal{H}}_{k(\delta)}^\varphi(E) \leq \sum_i \sum_j \varphi(n_s^{-k_i}) \leq \sum_i \sum_j \varphi(|U_i|) \leq C_2 \sum_i \varphi(|U_i|).$$

Taking infimum over covers $\{U_i\}$ and letting $\delta \rightarrow 0$ give $\tilde{\mathcal{H}}^\varphi(E) \leq C_2 \mathcal{H}^\varphi(E)$.

The proof is finished by taking $C = \max\{C_1, C_2\}$. □

Proposition B.3. *Let φ be a gauge function and $E \subset \mathbb{T}^d$. If $0 < \tilde{\mathcal{H}}^\varphi(E) < \infty$, then for $\tilde{\mathcal{H}}^\varphi$ -a.e. $x \in E$,*

$$\limsup_{k \rightarrow \infty} \frac{\tilde{\mathcal{H}}^\varphi(E \cap Q_k(x))}{\varphi(n_s^{-k})} = 1.$$

Proof. For $m, n \in \mathbb{N}$, define

$$F_m^n = \left\{ x \in E : \tilde{\mathcal{H}}^\varphi(E \cap Q_\ell(x)) \leq \frac{n-1}{n} \varphi(n_s^{-\ell}) \text{ for all } \ell \geq m \right\}.$$

To prove $\limsup_{k \rightarrow \infty} \tilde{\mathcal{H}}^\varphi(E \cap Q_k(x)) / \varphi(n_s^{-k}) \geq 1$, it suffices to prove $\tilde{\mathcal{H}}^\varphi(F_m^n) = 0$. For $m' \geq m$, let $\{Q_i\}$ be a cover of F_m^n such that $Q_i \in \mathcal{Q}_{k_i}$, $k_i \geq m'$ and $Q_i \cap F_m^n \neq \emptyset$. Then for each Q_i there exists $x_i \in F_m^n$ such that $Q_i = Q_{k_i}(x_i)$. Hence

$$\begin{aligned} \sum_i \varphi(n_s^{-k_i}) &\geq \frac{n}{n-1} \sum_i \tilde{\mathcal{H}}^\varphi(E \cap Q_{k_i}(x_i)) && \text{(by } x_i \in F_m^n) \\ &\geq \frac{n}{n-1} \sum_i \tilde{\mathcal{H}}^\varphi(F_m^n \cap Q_{k_i}(x_i)) && \text{(by } F_m^n \subset E) \\ &\geq \frac{n}{n-1} \tilde{\mathcal{H}}^\varphi(F_m^n) && \text{(by } \{Q_{k_i}(x_i)\}_i \text{ covers } F_m^n). \end{aligned}$$

Taking infimum over covers $\{Q_i\}$ and letting $m' \rightarrow \infty$ give

$$\tilde{\mathcal{H}}^\varphi(F_m^n) \geq \frac{n}{n-1} \tilde{\mathcal{H}}^\varphi(F_m^n).$$

Combining this with $\tilde{\mathcal{H}}^\varphi(F_m^n) \leq \tilde{\mathcal{H}}^\varphi(E) < \infty$ and $n/(n-1) > 1$ shows $\tilde{\mathcal{H}}^\varphi(F_m^n) = 0$.

For $t > 1$, define

$$F = \left\{ x \in E : \limsup_{k \rightarrow \infty} \frac{\tilde{\mathcal{H}}^\varphi(E \cap Q_k(x))}{\varphi(n_s^{-k})} > t \right\}.$$

To prove $\limsup_{k \rightarrow \infty} \tilde{\mathcal{H}}^\varphi(E \cap Q_k(x)) / \varphi(n_s^{-k}) \leq 1$, it suffices to prove $\tilde{\mathcal{H}}^\varphi(F) = 0$. By $\tilde{\mathcal{H}}^\varphi(F) < \infty$ and the outer regularity of $\tilde{\mathcal{H}}^\varphi|_E$, for $\varepsilon > 0$ there exists an open set $U \supset F$ such that

$$(B.1) \quad \tilde{\mathcal{H}}^\varphi(E \cap U) \leq \tilde{\mathcal{H}}^\varphi(F) + \varepsilon.$$

Let $N \in \mathbb{N}$. Then for each $x \in F$, there exists $k_x \geq N$ such that

$$\tilde{\mathcal{H}}^\varphi(E \cap Q_{k_x}(x)) > t\varphi(n_s^{-k_x}) \quad \text{and} \quad Q_{k_x}(x) \subset U.$$

Then $\{Q_{k_x}(x)\}_{x \in F}$ is a cover of F . By the net structure of approximate cubes, there exists a disjoint subcover $\Gamma = \{Q_i\}$ of $\{Q_{k_x}(x)\}_{x \in F}$ such that for each $Q_i \in \Gamma$ with $Q_i \in \mathcal{Q}_{k_i}$,

$$(B.2) \quad \tilde{\mathcal{H}}^\varphi(E \cap Q_i) > t\varphi(n_s^{-k_i}) \quad \text{and} \quad Q_i \subset U.$$

Hence

$$\begin{aligned} \tilde{\mathcal{H}}^\varphi(F) + \varepsilon &\geq \tilde{\mathcal{H}}^\varphi(E \cap U) && \text{(by (B.1))} \\ &\geq \sum_i \tilde{\mathcal{H}}^\varphi(E \cap Q_i) && \text{(by } Q_i \text{ being disjoint and (B.2))} \\ &> t \sum_i \varphi(n_s^{-k_i}) && \text{(by (B.2))} \\ &\geq t \tilde{\mathcal{H}}^\varphi(F) && \text{(by } \Gamma \text{ being a cover of } F). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, we conclude from $t > 1$ and $\tilde{\mathcal{H}}^\varphi(F) \leq \tilde{\mathcal{H}}^\varphi(E) < \infty$ that $\tilde{\mathcal{H}}^\varphi(F) = 0$. \square

We remark that [Proposition B.3](#) can be alternatively obtained by applying [\[10, Corollary\]](#) with $\Phi_k = \bigcup_{\ell \geq k} \mathcal{Q}_\ell$ for $k \in \mathbb{N}$ and $g(Q) = \varphi(n_s^{-k})$ for $Q \in \mathcal{Q}_k$.

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