A note on the thermodynamic formalism

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May 8, 2023

Abstract

We survey some results about the thermodynamic formalism, especially the extensions to subadditive potentials and weighted measure-theoretic entropies. As an application, we review the mutifractal analysis for Lyapunov exponents with respect to subadditive potentials.

1 Introduction

What is the *thermodynamic formalism*? In dynamic system and ergodic theory, it is a topic that studies the relations between the measure-theoretic entropy and topological pressure considering the Lyapunov exponents. For details about these classical dynamic quantities, please refer to [16] .

We call (X,T) a *topological dynamical system* (TDS) if X is a compact metric space and $T\colon X \to X$ is a continuous transformation. Let $C(X)$ be the space of continuous functions on X. Let $\mathcal{M}(X)$ be the set of Borel probability measure and $\mathcal{M}_T(X) \subset \mathcal{M}(X)$ be the set of T-invariant measures. The *classical thermodynamic formalism* (CTF) reveals the natural relations among the following quantities (see subsubsection 2.2.3).

- The measure-theoretic entropy $h\colon \mathcal{M}_T(X) \to [0,\infty]$.
- The Lyapunov exponent $\Phi_f\colon \mathcal{M}_T(X) \to (-\infty,\infty)$.
- The topological pressure $P\colon C(X) \to (\infty,+\infty]$.

Assuming the finiteness of topological entropy and the upper semi-continuity of measure-theoretic entropy, the CTF shows that

$$(-h) \xrightarrow{*} P \xrightarrow{*} (-h) \tag{1.1}$$

where $*$ means taking the *Legendre transform* of a convex function (see subsection 2.2). In particular, the equation $P = (-h)^*$ is called the *variational principle*. Under further conditions, the CTF also gives some properties of the *equilibrium states* which constitutes

$$\left\{\mu \in \mathcal{M}_T(X)\colon \int f\, d\mu + h(\mu) = P(f)\right\}.$$

For a complete account of the classical thermodynamic formalism, please refer to [16]. Some historical progress is outlined in Appendix B.

Recent research aims to extend the above results to some general settings or give conditions to assure the related results. The CTF hints at the good theorems, and to some extent at their proofs. Motivated by the dimension theory in dynamic systems and iterated function systems (IFS), there are two major directions for the extension: *subadditive thermodynamic formalism* and *weighted thermodynamic formalism*.

What is the subadditive thermodynamic formalism? By generalizing the additive potentials (in the definition of Lyapunov exponent and topological pressure) to the subadditive ones, we step into the study of subadditive thermodynamic formalism. Why it matters? For instance, by Bowen's equation [5] and Falconer's result [7], the dimensions of some conformal repellers and typical self-affine sets are the zeros of some topological pressures defined by some non-additive potentials. Meanwhile, the local structure of some self-similar measures with overlaps [12] can be expressed as products of matrices which induces a subadditive potential.

What is the weighted thermodynamic formalism? It is about the investigation of a formalism where the measure-theoretic entropy is defined with respect to a chain of factor maps instead of a single TDS. Why it matters? It is motivated by Kenyon-Peres's work [13] for the Hausdorff dimension of invariant sets under diagonal toral endomorphisms. The distinct scaling ratios along different directions lead us to consider a chain of factor maps, thus the weighted thermodynamic formalism.

What is the *multifractal analysis*? Multifractal analysis studies the size of level sets of some interesting pointwise quantity. In Section 4, the quantity is the pointwise Lyapunov exponent (see (4.2)) and the level sets are measured by the topological entropy (see (4.4)). The *multifractal formalism* usually aims to express the sizes of level sets as the Legendre transform of another function. During the study, one important tool is the thermodynamic formalism.

This article is organized as follows. In Section 2, we introduce some necessary definitions and briefly review the classical thermodynamic formalism. Based on [6, 9], some results about subadditive thermodynamic formalism are given in Section 3. As an application, we conduct the multifractal analysis for Lyapunov exponents in Section 4 according to [10]. In Section 5, following [9, 11], we establish the weighted thermodynamic formalism. For clearity and simplicity, the results are presented in symbolic dynamics. However, many of them have natural extensions to topological dynamics. Finally, some questions are raised in Section 6.

2 Preliminaries

2.1 Dynamic quantities

Let (X, T) be a TDS in this subsection. We will define the measure-theoretic entropy, Lyapunov exponents, and topological pressures. Some properties of them are given.

2.1.1 Measure-theoretic entropy

Let $\mu \in \mathcal{M}_T(X)$. For a finite partition ξ of X, define the *partition entropy*

$$H(\mu, \xi) := \sum_{A\in\xi} -\mu(A)\log\mu(A)$$

and the *dynamic partition entropy*

$$h(\mu,\xi) := \lim_{n\to\infty}\frac{1}{n}H\left(\mu, \bigvee_{i=0}^{n-1}T^{-i}\xi\right) = \inf_{n\in\mathbb{N}}\frac{1}{n}H\left(\mu, \bigvee_{i=0}^{n-1}T^{-i}\xi\right).$$

Then the *measure-theoretic entropy* $h\colon \mathcal{M}_T(X)\to[0,\infty]$ is

$$h(\mu) := \sup\{h(\mu,\xi)\colon \xi \text{ is a finite partition of } X\} \tag{2.1}$$

Next we give some properties of $\mu\mapsto h(\mu)$.

Proposition 2.1. *The map $\mu\mapsto h(\mu)$ is affine.*

Proof. See e.g. [16, Theorem 8.1]. □

The following proposition shows the upper semi-continuity of dynamic partition entropy under the Birkhoff limit.

Proposition 2.2. *Let $\{\eta_n\}\subset\mathcal{M}_T(X)$. Define $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}T^i\mu_n$ for $n\in\mathbb{N}$. Let μ be any w-∗ limit point of $\{\mu_n\}$. Let ξ be a finite partition consisting of sets with μ-null boundaries, that is, $\mu(\partial A)=0$ for $A\in\xi$. Then*

$$\limsup_{n\to\infty}\frac{1}{n}H\left(\eta_n, \bigvee_{i=0}^{n-1}T^{-i}\xi\right)\le h(\mu,\xi)$$

This is directly deduced from the next lemma due to Misiurewicz, see e.g., [16, Theorem 9.10].

Lemma 2.3. *Let $\mu\in\mathcal{M}_T(X)$ and ξ be a finite partition. Let $k\in\mathbb{N}$. Then for $n\ge 2k$,*

$$\frac{1}{n}H\left(\mu, \bigvee_{i=0}^{n-1}T^{-i}\xi\right)\le\frac{1}{k}H\left(\frac{1}{n}\sum_{i=0}^{n-1}T^i\mu, \bigvee_{i=0}^{k-1}T^{-i}\xi\right)+O\left(\#\xi\frac{k}{n}\right)$$

Proof. It is proved by the subadditiviy of the partition entropy and the next two combinatoric facts. For $j=0,\dots,k-1$, by Euclidean algorithm we have

$$n-j=q_jk+r_j$$

for some $q_j\ge 1$ and $0\le r_j<k$. Then

$$\{0,\dots,n-k\}=\bigsqcup_{j=0}^{k-1}\{j+\ell k\colon \ell=0,\dots,q_j-1\}$$

and for $j=0,\dots,k-1$,

$$\{0,\dots,n-1\}=\bigsqcup_{\ell=0}^{q_j-1}\{j+\ell k+i\colon i=0,\dots,k-1\}\bigsqcup\{0,\dots,j-1,j+q_jk,\dots,n-1\}.$$

□

2.1.2 Lyapunov exponent

We say that a sequence $\Phi = \{\log \phi_n\}$ of functions on X is a *subadditive potential* if each function $\phi_n \colon X \to [0,\infty)$ is a continuous function such that

$$\phi_{n+m}(x) \lesssim \phi_n(x)\phi_m(T^n x) \quad \text{for } x \in X.$$

If $\phi_n(x) = \exp(\sum_{i=0}^{n-1} f(T^i x))$ for some $f \in C(X)$, then Φ is called *additive*. For convenience, we denote the collection of all subadditive potentials on X by $O(X)$.

The *Lyapunov exponent* $\Phi \colon \mathcal{M}_T(X) \to [-\infty, \infty)$ is

$$\Phi(\mu) := \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu \quad \text{for } \mu \in \mathcal{M}_T(X) \tag{2.2}$$

where the limit exists by the subadditivity. In particular, for $f \in C(X)$, define $\Phi_f = \{\exp(\sum_{i=0}^{n-1} f(T^i x))\}$, then

$$\Phi_f(\mu) = \int f \, d\mu. \tag{2.3}$$

Next we give some properties of $\mu \mapsto h(\mu)$.

Proposition 2.4. *The map $\mu \mapsto \Phi(\mu)$ is affine.*

The following proposition shows some upper semi-continuities of $\mu \mapsto \Phi(\mu)$.

Proposition 2.5. *Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Then*

(i) The map $\mu \mapsto \Phi(\mu)$ is upper semi-continuous in w-$*$ topology.

(ii) Let $\{\eta_n\} \subset \mathcal{M}_T(X)$. Define $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i \mu_n$ for $n \in \mathbb{N}$. Let μ be any w-$*$ limit point of $\{\mu_n\}$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\eta_n \le \Phi(\mu).$$

Proof. (i) follows directly from the subadditivity while (ii) results from Lemma 2.6. □

Lemma 2.6. *Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Let $k \in \mathbb{N}$. Then for $n \ge 3k$,*

$$\frac{1}{n}\sum_{i=1}^{n} \log \phi_i(x) \le \frac{1}{k} \sum_{i=1}^{n-k} \log \phi_k(T^i x) + O\left(\frac{k}{n}\right).$$

Proof. It is proved by the subadditivity of Φ and the combinatoric facts in the proof of Lemma 2.3. □

2.1.3 Topological pressure

Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Denote the metric on X by d. For $n \in \mathbb{N}$, define

$$d_n(x,y) := \max_{0\le i\le n-1} d(T^i x, T^i y) \quad \text{for } x,y\in X.$$

Fix any discretization constant $\varepsilon > 0$. A subset $E \subset X$ is called a (n,ε)-*separated set* if $d_n(x,y) \ge \varepsilon$ for distinct points $x,y \in E$. Define

$$P(\Phi,\varepsilon,n) := \sup\left\{\sum_{x\in E}\phi_n(x)\colon E \text{ is a } (n,\varepsilon)\text{-separated set of } X\right\}$$

and

$$P(\Phi,\varepsilon) := \limsup_{n\to\infty}\frac{1}{n}\log P(\Phi,\varepsilon,n). \tag{2.4}$$

The *topological pressure* $P\colon O(X)\to[-\infty,\infty]$ is defined by

$$P(\Phi) := \lim_{\varepsilon\to 0} P(\Phi,\varepsilon).$$

In particular, for $f \in C(X)$, define $\Phi_f = \{\exp(\sum_{i=0}^{n-1} f(T^i x))\}$ and $P\colon C(X)\to[\infty,\infty]$ by

$$P(f) := P(\Phi_f). \tag{2.5}$$

2.2 Legendre transform

Let X and X^* be real locally convex topological vector spaces. Let $\langle\cdot,\cdot\rangle\colon X\times X^*\to\mathbb{R}$ be a separately continuous bilinear function, that is, for x, x^*, the maps

$$\langle\cdot,x^*\rangle\colon X\to\mathbb{R},\qquad \langle x,\cdot\rangle\colon X^*\to\mathbb{R}$$

are continuous, and for $x,y\in X$, $x^*,y^*\in X^*$, $\alpha\in\mathbb{R}$,

$$\langle \alpha x + y, x^*\rangle = \alpha\langle x,x^*\rangle + \langle y,x^*\rangle,\quad \langle x,\alpha x^*+y^*\rangle = \alpha\langle x,x^*\rangle+\langle x,y^*\rangle.$$

Let $f\colon X\to[-\infty,\infty]$ be a extended-real valued function. The *convex conjugate* of f is $f^*\colon X^*\to[-\infty,\infty]$ defined by

$$f^*(x^*) := \sup\{\langle x,x^*\rangle - f^*(x)\colon x\in X\} \qquad \text{for } x^*\in X^*. \tag{2.6}$$

We call f a *convex function* if its *epigraph*

$$\operatorname{epi}(f) := \{(x,\alpha)\colon x\in X, \alpha\in\mathbb{R},\, \alpha\ge f(x)\}$$

is a convex set in $X\oplus\mathbb{R}$. In particular, when f is real-valued, then f is a convex function if and only if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \quad \text{for } x,y\in X, \lambda\in(0,1).$$

The *effective domain* of f is

$$\mathrm{dom}(f) := \{x \in X \colon f(x) < \infty\}.$$

A convex function f is said to be *proper* if $\mathrm{dom}(f)$ is not empty and contains no vertical lines, i.e. if $f(x) < \infty$ for at least one x and $f(x) > -\infty$ for all x. By convention, we will view every real-valued convex function f on $E \subset X$ as a convex function on X by setting $f = \infty$ on $X \setminus E$.

Definition 2.7 (Legendre transform)**.** Let f be a proper convex function. The *Legendre transform* of f is the convex conjugate f^* defined in (2.6).

A point $x^* \in X^*$ is called a *subgradient* of a convex function f at x if

$$f(x+h) \geq \langle h, x^* \rangle + f(x) \qquad \text{for all } h \in X.$$

The collection of all subgradients at x is called *subdifferential* at x and denoted by $\partial f(x)$. Let $K \subset X$ be a convex set. We denote the extreme points of K by $\mathrm{ex}(K)$. The Krein-Milman theorem implies that $\mathrm{ex}(K) \neq \emptyset$ if K is non-empty and compact. For notational simplicity, we write $\partial^e f(x) := \mathrm{ex}(\partial f(x))$.

There are some usual examples of X and X^*.

- Let $X = X^* = \mathbb{R}^d$ and $\langle x, x^* \rangle := x \cdot x^*$ be the standard inner product.
- Let K be a compact metric space. Let $X := C(K)$ equipped with the sup norm. Let $X^* := (C(K))^*$ be the dual space of X consisting of Radon measures by Riesz representation theorem. Endow X^* with the w-$*$ topology induced by X. Finally, define
$$\langle f, \mu \rangle := \int f \, d\mu \qquad \text{for } f \in X,\ \mu \in X^*.$$

2.2.1 Basic example

Below is an example of Legendre transform by direct computation.

- Let $f \colon \mathbb{R} \to [-\infty, \infty]$ be
$$f(x) := \begin{cases} x^2 & \text{if } x \in (-1, 1) \\ 2 & \text{if } x = \pm 1 \\ \infty & \text{otherwise.} \end{cases}$$
- The convex conjugate of f is $f^* \colon \mathbb{R} \to [-\infty, \infty]$ given by
$$f^*(x^*) = \begin{cases} -x^* - 1 & \text{if } x^* \leq -2 \\ \frac{(x^*)^2}{4} & \text{if } x^* \in (-2, 2) \\ x^* - 1 & \text{if } x^* \geq 2. \end{cases}$$

- The double convex conjugate of f is $f^{**} \colon \mathbb{R} \to [-\infty, \infty]$ given by

$$f^{**}(x^{**}) \begin{cases} (x^{**})^2 & \text{if } x \in [-1,1] \\ \infty & \text{otherwise.} \end{cases}$$

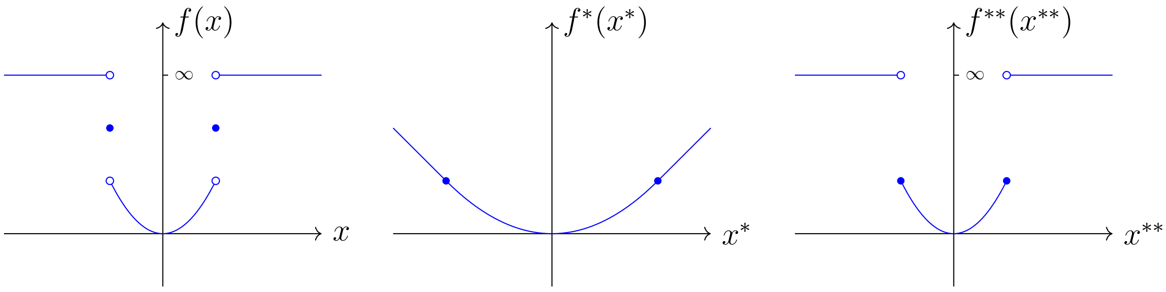

Figure 1: An example of convex conjugates

Notice that

$$f(x) = f^{**}(x) \quad \text{if } |x| \neq 1 \quad \text{and} \quad f(x) > f^{**}(x) \quad \text{if } |x| = 1.$$

Hence $f = f^{**}$ fails at the discontinuities of f. This inspires us to assume the upper semi-continuity $\mu \mapsto h(\mu)$ in the Fenchel-duality part of thermodynamic formalism, for example (4.9).

2.2.2 Bernoulli thermodynamic formalism

In this example, we show the convex-conjugate relations in the following quantities. Let $\Delta^{d-1} \subset \mathbb{R}^d$ denote the $(d-1)$-dimensional simplex, that is,

$$\Delta^{d-1} := \{(p_1, \dots, p_d) \in \mathbb{R}^d \colon \sum_{i=1}^{d} p_i = 1,\, p_i \geq 0\}.$$

- The Shannon entropy $H \colon \Delta^{d-1} \to [0, \log d]$ is

$$H(\mathbf{p}) := \sum_{i=1}^{d} -p_i \log p_i \quad \text{for } \mathbf{p} = (p_1, \dots, p_d) \in \Delta^{d-1}.$$

- Let $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$. The linear function $\Phi_{\mathbf{a}} \colon \mathbb{R}^d \to \mathbb{R}$ is

$$\Phi_{\mathbf{a}}(\mathbf{p}) := \langle \mathbf{a}, \mathbf{p} \rangle = \sum_{i=1}^{d} p_i a_i \quad \text{for } \mathbf{p} = (p_1, \dots, p_d) \in \mathbb{R}^d.$$

- The partition function $Z \colon \mathbb{R}^d \to \mathbb{R}$ is

$$Z(\mathbf{a}) := \log \sum_{i=1}^{d} \exp(a_i) \quad \text{for } \mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d.$$

Proposition 2.8 (Bernoulli thermodynamic formalism)**.** *In the above notation, we have*

$$(-H) \xrightarrow{*} Z \xrightarrow{*} (-H).$$

This means

$$Z(\mathbf{a}) = \max\left\{\langle \mathbf{a}, \mathbf{p}\rangle + H(\mathbf{p}) \colon \mathbf{p} \in \Delta^{d-1}\right\} \qquad \text{for } \mathbf{a} \in \mathbb{R}^d$$

and

$$-H(\mathbf{p}) = \max\left\{\langle \mathbf{a}, \mathbf{p}\rangle - Z(\mathbf{a}) \colon \mathbf{a} \in \mathbb{R}^d\right\} \qquad \text{for } \mathbf{p} \in \Delta^{d-1}.$$

Moreover, the maximizer for each equation above is unique.

Proposition 2.8 follows directly from the next lemma.

Lemma 2.9 (Generalized Gibbs inequality)**.** *Let* $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n_{\geq 0}$. *Then*

$$\sum_i -a_i \log a_i + \sum_i a_i \log b_i \leq -(\sum_i a_i) \log(\sum_i a_i) + (\sum_i a_i) \log(\sum_i b_i)$$

where the equality holds if and only if $\mathbf{a} \parallel \mathbf{b}$, *that is,*

$$\frac{a_i}{\sum_j a_j} = \frac{b_i}{\sum_j b_j} \quad \text{for } i = 1, \dots, n.$$

Proof. It is a direct result of the strict convexity of $x \mapsto \log x$. □

2.2.3 Classical thermodynamic formalism

Let (X, T) be a TDS where the topological entropy is finite and the measure-theoretic entropy is upper semi-continuous. The classical thermodynamic formalism reveals the convex-conjugate relations between the following dynamic quantities.

- The measure-theoretic entropy $h \colon \mathcal{M}_\sigma(X) \to [0, \infty)$ is defined in (2.1).
- The Lyapunov exponent with respect to $f \in C(X)$ is $\Phi_f \colon \mathcal{M}(X) \to \mathbb{R}$ defined in (2.3).
- The topological pressure $P \colon C(X) \to (\infty, +\infty]$ is defined in (2.5).

Then the negative measure-theoretic entropy and the topological pressure are convex conjugates to each other.

Theorem 2.10 (Fenchel duality in CTF)**.** *Let* (X, T) *be a TDS where the topological entropy is finite and the measure-theoretic entropy is upper semi-continuous. Then*

$$(-h) \xrightarrow{*} P \xrightarrow{*} (-h).$$

This means

$$P(f) = \sup\left\{\langle f, \mu\rangle + h(\mu) \colon \mu \in \mathcal{M}_T(X)\right\} \qquad \text{for } f \in C(X)$$

and

$$h(\mu) = \inf\left\{P(f) - \langle f, \mu\rangle \colon f \in C(X)\right\} \qquad \text{for } \mu \in \mathcal{M}_T(X).$$

Theorem 2.10 is the classical thermodynamic formalism without refering to the equilibrium states (see [16]). On the other hand, as the main result in [4], the next theorem gives a condition on the uniqueness of the equilibrium state. For the details, please refer to [4].

Theorem 2.11 (Bowen's unique equilibrium state). *Let (X, T) be a TDS. Suppose that X satisfies specification and T is expansive. Then for $f \in C(X)$ with bounded distorsion, there is a unique equilibrium state such that*

$$\int f\, d\mu + h(\mu) = P(f).$$

The proof of Theorem 2.11 contains the an important criterion due to Parry [15] for the absolute continuity between measures.

Lemma 2.12. *Let (X, T) be a TDS. Suppose that ξ is a strong generator of the Borel σ-algebra on X. Let $\eta, \mu \in \mathcal{M}(X)$. If*

$$\sum_{A \in \bigvee_{i=0}^{n-1} T^{-i}\xi} -\eta(A) \log \eta(A) + \eta(A) \log \mu(A) \geq O(1) \tag{2.7}$$

for all $n \in \mathbb{N}$, then $\eta \ll \mu$.

Proof. It follows from the approximation of strong generators and Lemma 2.9. □

2.3 Subshifts

Let $\mathcal{A}$ be a finite set called the *alphabet*. We endow $\mathcal{A}^{\mathbb{N}}$ with the canonical metric

$$d(x, y) := 2^{-|x \wedge y|} \quad \text{for } x, y \in \mathcal{A}^{\mathbb{N}}$$

where $x \wedge y$ denotes the common prefix of x, y and $|I| = n$ for $I \in \mathcal{A}^n, n \in \mathbb{N}$. Then $\mathcal{A}^{\mathbb{N}}$ is a compact metric space. The (left) shift map $\sigma\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by $\sigma(x) := (x_{i+1})$ for $x = (x_i) \in \mathcal{A}^{\mathbb{N}}$. The TDS $(\mathcal{A}^{\mathbb{N}}, \sigma)$ is called the (one-sided) *fullshift* over $\mathcal{A}$.

Let X be a closed subset of $\mathcal{A}^{\mathbb{N}}$ such that $\sigma X \subset X$. Then (X, σ) is a *subshift*, or simply call X a subshift. For $n \in \mathbb{N}$, define

$$\mathcal{L}_n(X) := \{I \in \mathcal{A}^n \colon I = x_1 \dots x_n \text{ for some } (x_i)_{i=1}^{\infty} \in X\}$$

and

$$\mathcal{L}(X) := \bigcup_{n=1}^{\infty} \mathcal{L}_n(X) \bigcup \{\varnothing\}$$

where $\varnothing$ denotes the empty word.

3 Subadditive theormodynamic formalism

3.1 Subadditive variational principle

We begin with a relativized version of variational principle which will be used in Section 5.

Theorem 3.1 (Relativized variational principle). *Let $\pi\colon X \to Y$ be an one-block factor map between subshifts X, Y. Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Let $\nu \in \mathcal{M}_\sigma(Y)$. Then*

$$\max\{\Phi(\mu) + h(\mu)\colon \mu \in \mathcal{M}_\sigma(X),\ \pi\mu = \nu\} = \Psi(\nu) + h(\nu) \tag{3.1}$$

where $\Psi = \{\log \psi_n\}$ is a subadditive potential on Y defined by

$$\psi_n(y) = \sum_{I\in\mathcal{L}_n(X)\colon \pi(I)=y|n} \sup_{x\in I} \phi_n(x) \quad \text{for } y \in Y \text{ and } n \in \mathbb{N}.$$

Proof. The upper bound in (3.1) follows from Lemma 2.9 directly. On the other hand, by Gibbs construction we can obtain an invariant measure μ with $\Phi(\mu) + h(\mu)$ maximal. Specifically, Lemma 2.9 hints a sequence of discrete measures η_n such that $H(\eta_n, \mathcal{L}_n(X)) + \int \log \phi_n \, d\eta_n$ maximal in the fiber $\pi^{-1}\left(\nu|_{\mathcal{L}_n(X)}\right)$. Then the Birkhoff limit of $\{\eta_n\}$ gives an invariant measure μ with $\pi\mu = \nu$. Finally Proposition 2.5 and Proposition 2.2 assure that $\Phi(\mu) + h(\mu)$ is maximal. □

When $Y = \{0\}$, Theorem 3.1 implies the subadditive variational principle.

Theorem 3.2 (Subadditive variational principle). *Let X be a subshift and $\Phi = \{\log \phi_n\}$ be a subadditive potential. Then*

$$\max\{\Phi(\mu) + h(\mu)\colon \mu \in \mathcal{M}_\sigma(X)\} = P(\Phi)$$

where

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{I\in\mathcal{L}_n(X)} \sup_{x\in I} \phi_n(x).$$

The subadditive variational principle is established for general TDS in [6, Theorem 1.1].

3.2 Unique subadditive equilibrium state

Definition 3.3 (Weak specification). Let X be a subshift. We say that X satisfies *weak specification* if there exists $p \in \mathbb{N}$ such that for every $I, J \in \mathcal{L}(X)$, there exists $W \in \mathcal{L}(X)$ with $|W| \leq p$ so that $IWJ \in \mathcal{L}(X)$.

Let X be a subshift over a finite alphabet $\mathcal{A}$. Let $\phi\colon \mathcal{L}(X) \to [0, \infty)$ be a function such that $\phi(IJ) \lesssim \phi(I)\phi(J)$ for $I, J \in \mathcal{L}(X)$. There is a subadditive potential $\Phi = \{\log \phi_n\}$ *induced* from ϕ where

$$\phi_n(x) := \phi(x|n) \quad \text{for } n \in \mathbb{N}.$$

We use $\Theta(X)$ to denote the collection of functions $\phi\colon \mathcal{L}(X) \to [0, \infty)$ such that

(1) There exists $W \in \mathcal{L}(X) \setminus \{\varnothing\}$ such that $\phi(W) > 0$.

(2) $\phi(IJ) \lesssim \phi(I)\phi(J)$ for all $IJ \in \mathcal{L}(X)$.

(3) There exists $p \in \mathbb{N}$ such that for any $I, J \in \mathcal{L}(X)$, there is $K \in \mathcal{L}(X)$ with $|W| \leq p$ such that $IWJ \in \mathcal{L}(X)$ and $\phi(IWJ) \gtrsim \phi(I)\phi(J)$.

Note that the constant function $\phi \equiv 1$ belongs to $\Theta(X)$ if X satisfies weak specification.

Theorem 3.4 (Unique subadditive equilibrium state)**.** *Let X be a subshift satisfying weak specification. Let $\phi \in \Theta(X)$ and $\Phi = \{\log \phi_n\}$ be the subadditive potential induced from ϕ. Then there is a unique equilibrium state μ of Φ, that is,*

$$\Phi(\mu) + h(\mu) = P(\Phi).$$

Moreover, μ is ergodic and satisfies

$$\mu(I) \approx \frac{\phi(I)}{\sum_{J \in \mathcal{L}_n(X)} \phi(J)} \approx \frac{\phi(I)}{\exp(nP(\Phi))} \quad \text{for } I \in \mathcal{L}_n(X)$$

and

$$\sum_{I \in \mathcal{L}_n(X)} -\mu(I) \log \mu(I) = nh(\mu) + O(1), \qquad \sum_{I \in \mathcal{L}_n(X)} -\mu(I) \log \phi(I) = n\Phi(\mu) + O(1)$$

The proof of Theorem 3.4 relies on Proposition 3.5 about a lower Gibbs property of some limit measure in the Gibbs construction. Let $\Omega(X)$ denote the collection of functions $\phi\colon \mathcal{L}(X) \to [0, \infty)$ such that

(1) $\sum_{I \in \mathcal{L}_n(X)} \phi(I) = 1$ for $n \in \mathbb{N}$.

(2) For $I \in \mathcal{L}(X)$, there exist $i, j \in \mathcal{A}$ such that $\phi(Ii) \gtrsim \phi(I)$ and $\phi(jI) \gtrsim \phi(I)$.

(3) There exists $p \in \mathbb{N}$ such that for any $I, J \in \mathcal{L}(X)$, there is $W \in \mathcal{L}(X)$ with $|W| \leq p$ such that $IWJ \in \mathcal{L}(X)$ and $\phi(IWJ) \gtrsim \phi(I)\phi(J)$.

The next proposition is a result of weak specification.

Proposition 3.5. *Let X be a subshift satisfying weak specification and $\phi \in \Omega(X)$. Fix any $x_I \in I$ for $I \in \mathcal{L}(X)$. For $n \in \mathbb{N}$, define*

$$\eta_n := \sum_{I \in \mathcal{L}_n(X)} \delta_{x_I} \phi(I)$$

and

$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \sigma^i \eta_n.$$

Then $\mu_n \xrightarrow{w\text{-}*} \mu$ as $n \to \infty$ for some $\mu \in \mathcal{M}_\sigma(X)$. Moreover, μ is ergodic and

$$\mu(I) \approx \phi^*(I) \gtrsim \phi(I) \quad \text{for } I \in \mathcal{L}(X)$$

where

$$\phi^*(I) := \sup_{m,n\geq 0} \phi_{m,n}(I)$$

and for $m, n \geq 0$,

$$\phi_{m,n}(I) := \sum_{I_1\in\mathcal{L}_m(X), I_2\in\mathcal{L}_n(X):\, I_1 I I_2\in\mathcal{L}(X)} \phi(I_1 I I_2).$$

Proof. By weak specification, we can show firstly for $I \in \mathcal{L}(X)$,

$$\phi^*(I) \approx \phi_{m,n}(I)$$

for m, n large, and secondly for $I, J \in \mathcal{L}(X)$,

$$\liminf_{n\to\infty} \sum_{k=0}^{p} \sum_{W\in\mathcal{L}_{n+k}(X)} \phi(IWJ) \gtrsim \phi(I)\phi(J).$$

The proof is completed by the above facts. □

Proof of Theorem 3.4. Since $\phi \in \Theta(X)$, it is proved that $Z_{m+n} \approx Z_m Z_n$ for $m, n \in \mathbb{N}$ where $Z_k := \sum_{I\in\mathcal{L}_k(X)} \phi(I)$. By normalizing with Z_n, we have $\widetilde{\phi} \in \Theta(X)$ where

$$\widetilde{\phi}(I) := \frac{\phi(I)}{Z_{|I|}} \approx \frac{\phi(I)}{\exp(nP(\Phi))} \quad \text{for } I \in \mathcal{L}(X).$$

Let $I \in \mathcal{L}_n(X)$. The subadditvity of $\widetilde{\phi}$ implies $\widetilde{\phi}^*(I) \approx \widetilde{\phi}(I)$. Therefore, Proposition 3.5 gives an ergodic measure μ such that

$$\mu(I) \approx \widetilde{\phi}^*(I) \approx \widetilde{\phi}(I) \approx \frac{\phi(I)}{\exp(nP(\Phi))}. \tag{3.2}$$

Let η be any equilibrium state provided by Theorem 3.2. Then

$$\Phi(\mu) + h(\mu) = P(\Phi). \tag{3.3}$$

Hence

$$\begin{aligned} 0 &\geq \sum_{I\in\mathcal{L}_n(X)} -\eta(I)\log\eta(I) + \eta(I)\log\mu(I) \\ &\geq \sum_{I\in\mathcal{L}_n(X)} \Big(-\eta(I)\log\eta(I) + \eta(I)\log\phi(I)\Big) - nP(\Phi) + O(1) && \text{by (3.2)} \\ &\geq nh(\eta) + n\Phi(\eta) - nP(\Phi) + O(1) && \text{by subadditivity} \\ &= O(1) && \text{by (3.3).} \end{aligned}$$

This shows $\eta \ll \mu$ by Lemma 2.12. Thus $\eta = \mu$ since μ is ergodic. □

4 Multifractal analysis for Lyapunov exponents

Let X be a subshift. Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Define

$$\overline{\Phi} := \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in X} \phi_n(x) \qquad \in [-\infty, \infty)$$

where the limit exists by the subadditivity of Φ. To avoid triviality, we assume $\overline{\Phi} > -\infty$. In this section, we will study the following quantities.

- The measure-theoretic entropy $h\colon \mathcal{M}_\sigma(X) \to [0,\infty)$ is defined in (2.1).
- The Lyapunov exponent $\Phi\colon \mathrm{Dom}(\Phi) \to [-\infty,\infty)$ is defined by

  $$\Phi(\mu) := \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu \tag{4.1}$$

  where
  $$\mathrm{Dom}(\Phi) = \left\{ \mu \in \mathcal{M}(X)\colon \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu \text{ exists} \right\}.$$
  The subadditivity of Φ implies that $\mathcal{M}_\sigma(X) \subset \mathrm{Dom}(\Phi)$. For $x \in X$, let δ_x denote the Dirac measure at x. For $x \in X$ with $\delta_x \in \mathrm{Dom}(\Phi)$, define

  $$\Phi(x) := \Phi(\delta_x). \tag{4.2}$$

- The topological pressure $P\colon O(X) \to [-\infty, +\infty)$ is simplified to

  $$P(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{I\in\mathcal{L}_n(X)} \sup_{x\in I} \phi_n(x)$$

  where the limit exists by the subadditivity and $O(X)$ denotes the collection all subadditive potentials on X.

By (4.2), we can view Φ as a point function. There is a *multifractal decomposition* of X by

$$X = (X \setminus \mathrm{Dom}(\Phi)) \bigsqcup \bigsqcup_{\alpha\in[-\infty,\infty]} E(\alpha) \tag{4.3}$$

where $E(\alpha)$ denotes the level set, that is,

$$E(\alpha) := \{x \in X\colon \Phi(x) = \alpha\} \qquad \text{for } \alpha \in [-\infty, \infty].$$

To obtain a mutifractal spectrum for X, we need a set function to measure the size of level sets. Below we introduce the (Bowen's) *topological entropy* $h\colon 2^X \to [0,\infty)$. Let $E \subset X$. For $s \geq 0$ and $n \in \mathbb{N}$, define

$$\mathcal{M}_n^s(E) := \inf \left\{ \sum_i \exp(-sn_i)\colon \text{cylinders } \{I_i\} \text{ covers } E \text{ with } n_i = |I_i| \geq n \right\}$$

and

$$\mathcal{M}^s(E) := \lim_{n\to\infty} \mathcal{M}^s_n(E).$$

Then the *topological entropy* of E is

$$\begin{aligned} h(E) &:= \inf\{s \geq 0 \colon \mathcal{M}^s(E) = 0\} = \inf\{s \geq 0 \colon \mathcal{M}^s(E) < \infty\} \\ &= \sup\{s \geq 0 \colon \mathcal{M}^s(E) > 0\} = \sup\{s \geq 0 \colon \mathcal{M}^s(E) = \infty\}. \end{aligned} \tag{4.4}$$

Following [2], we can define the *topological entropy multifractal spectrum for the Lyapunov exponents* $\mathcal{F} \colon [-\infty, \infty] \to [0, \infty)$ by

$$\mathcal{F}(\alpha) := h(E(\alpha)) \quad \text{for } \alpha \in [-\infty, \infty]. \tag{4.5}$$

Here are some natural questions.

Q1: Does multifractal formalism hold for $\mathcal{F}$? Equivalently, can $\mathcal{F}$ expressed as the Legendre transform of another function?

Q2: Is there a variational principle between the topological entropy $h(E(\alpha))$ and measure-theoretic entropy $h(\mu)$ with $\mu(E(\alpha)) = 1$?

Before answering the above questions, we make some simplifications. Instead of doing convex analysis for $P \colon O(X) \to \mathbb{R}$ directly on the $O(X)$, we tend to analyze convexity along the one-parameter family of subadditive potentials $\{q\Phi\}_{q>0}$. Define the *reduced* topological pressure function $P \colon (0, \infty) \to \mathbb{R}$ by

$$P(q) := P(q\Phi) \quad \text{for } q > 0. \tag{4.6}$$

What is the convex conjugate of $q \mapsto P(q)$? A candidate can be found by Theorem 3.2,

$$\begin{aligned} P(q) &= P(q\Phi) \\ &= \sup_{\mu \in \mathcal{M}_\sigma(X)} \{q\Phi(\mu) + h(\mu)\} \\ &= \sup_{\alpha \in \Omega} \sup_{\Phi(\mu)=\alpha} \{q\Phi(\mu) + h(\mu)\} \\ &= \sup_{\alpha \in \Omega} \left\{ \alpha q + \sup_{\Phi(\mu)=\alpha} \{h(\mu)\} \right\} \end{aligned}$$

where $\Omega = \Phi(\mathcal{M}_\sigma(X))$. This leads us to define the *reduced* measure-theoretic entropy function $h \colon \Omega \to \mathbb{R}$ by

$$h(\alpha) := \{h(\mu) \colon \mu \in \mathcal{M}_\sigma(X) \text{ with } \Phi(\mu) = \alpha\} \quad \text{for } \alpha \in \Omega. \tag{4.7}$$

With the above simplifications, it is easy to formulate some conjectures about the answers to the above questions. Fortunately the conjectures are proved in [10].

Theorem 4.1 (Mulifractal formalism for Lyapunov exponents)**.** *Let X be a subshift and Φ be a subadditive potential on X. The function $q \mapsto P(q) := P(q\Phi)$ is continuous and convex on $(0, \infty)$. Moreover,* *(i) Let* $\alpha \in \partial^e P(q)$ *for some* $q > 0$*. Then* $E(\alpha) \neq \emptyset$ *and*

$$h(E(\alpha)) = \inf_{t>0}\{P(t) - \alpha t\}. \tag{4.8}$$

(ii) Let $\alpha \in (P'(0), P'(\infty))$*. Then*

$$h(\alpha) = \inf_{t>0}\{P(t) - \alpha t\} \tag{4.9}$$

where $h(\alpha)$ *is defined in* (4.7)*.*

Theorem 4.1 is a combination of Theorem 4.4 and Theorem 4.5. Note that the right hands of (4.8) and (4.9) are the negation of the Legendre transform of $P\colon (0,\infty) \to \mathbb{R}$. There is an immediate corollary.

Corollary 4.2. *Let* X *be a subshift and* Φ *be a subadditive potential on* X*. Let* $q > 0$*. If* $\mathcal{I}(q) = \{\mu_q\}$*, then* μ_q *is ergodic. Moreover,* $P'(q) = \Phi(\mu_q)$ *and*

$$h(E(P'(q))) = \inf_{t>0}\{P(t) - P'(q)t\} = h(P'(q)) = P(q) - P'(q)q.$$

A combination of Theorem 3.4 and Corollary 4.2 gives the following theorem.

Theorem 4.3. *Let* X *be a subshift satisfying weak specification. Let* $\phi \in \Theta(X)$ *and* $\Phi = \{\log \phi_n\}$ *be the subadditive potential induced from* ϕ*. Then the map* $q \mapsto P(q) := P(q\Phi)$ *is a convex differentiable function on* $(0,\infty)$*. Moreover, for* $q > 0$*,* $P'(q) = \Phi(\mu_q)$ *where* μ_q *is the unique equilibrium state of* $q\Phi$*, and*

$$h(E(P'(q))) = h(P'(q)) = \inf_{t>0}\{P(t) - P'(q)t\} = P(q) - P'(q)q.$$

4.1 Upper bound

Theorem 4.4. *Let* X *be a subshift and* Φ *be a subadditive potential on* X*. Then the map* $q \mapsto P(q) := P(q\Phi)$ *is a convex continuous function on* $(0,\infty)$*. Moreover, for* $\alpha \in (P'(0), P'(\infty))$*.*

(i) $h(E(\alpha)) \leq \inf_{t>0}\{P(t) - \alpha t\}$*.*

(ii) $h(\alpha) \leq \inf_{t>0}\{P(t) - \alpha t\}$*.*

Proof. First we show (i). Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Define

$$G(\alpha, n, \varepsilon) := \left\{ x \in X \colon \forall\, k \geq n,\ \frac{1}{k}\log \phi_k(x) \in (\alpha - \varepsilon, \alpha + \varepsilon) \right\}. \tag{4.10}$$

Write $F = G(\alpha, n, \varepsilon)$ for convenience. For $k \in \mathbb{N}$, denote

$$\mathcal{L}_k(F) := \{I \in \mathcal{L}_k(X) \colon I \cap F \neq \emptyset\}.$$

Then there exists some $x_I \in I \cap F$ for each $I \in \mathcal{L}_k(F)$.

Let $s > P(q)$. It follows from the definition of $P(q)$ that for all large k,

$$\begin{aligned} ks &> \log \sum_{I \in \mathcal{L}_k(X)} \sup_{x \in I} \phi_k^q(x) \\ &\geq \log \sum_{I \in \mathcal{L}_k(F)} \phi_k^q(x_I) \\ &\geq \log \#\mathcal{L}_k(F) + \sum_{I \in \mathcal{L}_k(F)} \frac{1}{\#\mathcal{L}_k(F)} \log \phi_k^q(x_I) && \text{by Lemma 2.9} \\ &\geq \log \#\mathcal{L}_k(F) + k(\alpha - \varepsilon)q && \text{by (4.10).} \end{aligned}$$

Hence $k[s - (\alpha - \varepsilon)q] \geq \log \#\mathcal{L}_k(F)$. Since $\mathcal{L}_k(F)$ is a cover of F with cylinders of length k,

$$\mathcal{M}_k^{s-(\alpha-\varepsilon)q}(F) \leq \#\mathcal{L}_k(F) \exp\left(-k[s - (\alpha - \varepsilon)q]\right) \leq \#\mathcal{L}_k(F) \exp(-\log \#\mathcal{L}_k(F)) = 1.$$

This implies the topological entropy

$$h(G(\alpha, n, \varepsilon)) \leq s - (\alpha - \varepsilon)q.$$

Since the topological entropy is monotone and σ-stable,

$$h(E(\alpha)) \leq h\left(\bigcup_{n=1}^{\infty} G(\alpha, n, \varepsilon)\right) = \sup_n h\left(G(\alpha, n, \varepsilon)\right) \leq s - (\alpha - \varepsilon)q.$$

Letting $\varepsilon \to 0$ finishes the proof.

Next we move to (ii). Let $t > 0$. By Theorem 3.2,

$$\begin{aligned} h(\alpha) &= \sup\left\{h(\mu) \colon \mu \in \mathcal{M}_\sigma(X),\ \Phi(\mu) = \alpha\right\} \\ &\leq \sup\left\{P(t) - t\Phi(\mu) \colon \mu \in \mathcal{M}_\sigma(X),\ \Phi(\mu) = \alpha\right\} \\ &= P(t) - \alpha t. \end{aligned}$$

Taking infimum with respect to $t > 0$ completes the proof. This is rather direct in the sense that Fenchel's inequality always holds. □

4.2 Lower bound

Theorem 4.5. *Let (X, σ) be a subshift and Φ be a subadditive potential on X. Then*

(i) Let $\alpha \in \partial^e P(q)$ for some $q > 0$. Then $E(\alpha) \neq \emptyset$ and $h(E(\alpha)) \geq \inf_{t>0}\{P(t) - \alpha t\}$.

(ii) Let $\alpha \in (P'(0), P'(\infty))$. Then $h(\alpha) \geq \inf_{t>0}\{P(t) - \alpha t\}$.

Theorem 4.5 relies on the next two lemmas.

Lemma 4.6. *For $q > 0$ and $\alpha \in \Omega$, the subset of equilibrium states with prescribed Lyapunov exponent*

$$\mathcal{I}(q, \alpha) := \{\mu \in \mathcal{I}(q) \colon \Phi(\mu) = \alpha\}$$

is compact convex if it is nonempty. Moreover, if $\alpha \in \partial^e P(q)$, then $\mathcal{I}(q, \alpha) \neq \emptyset$ and

$$\operatorname{ex}(\mathcal{I}(q, \alpha)) \subset \operatorname{ex}(\mathcal{M}_\sigma(X)).$$

Proof. The convexity follows from the affinity of Φ. To justify the compactness, it suffices to prove the closedness. Let μ be a closure point of $I(q,\alpha)$ in w-$*$ topology. By metrizability, there exists $(\mu_n) \in \mathcal{I}(q,\alpha)$ such that $\mu_n \xrightarrow{w-*} \mu$ as $n \to \infty$. The upper semi-continuity of $\nu \mapsto h(\nu)$ and $\nu \mapsto \Phi(\nu)$ implies

$$P(q) \geq q\Phi(\mu) + h(\mu) \geq \lim_{n\to\infty} (q\Phi(\mu_n) + h(\mu_n)) = \alpha q + \lim_{n\to\infty} h(\mu_n) = P(q)$$

which forces that $\Phi(\mu) = \alpha$ and $h(\mu) = P(q) - \alpha q$. Hence $\mu \in \mathcal{I}(q,\alpha)$.

Let $\alpha \in \partial^e P(q)$. Then $\mathcal{I}(q,\alpha) \neq \emptyset$ since $\partial P(q) = \Phi(\mathcal{I}(q))$. By Krein-Milman theorem, there exists some $\mu \in \mathrm{ex}(\mathcal{I}(q,\alpha))$. Suppose

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2$$

for some $\mu_1, \mu_2 \in \mathcal{M}_\sigma(X)$ and $\lambda \in (0,1)$. By the linearity and extremality, we have $\mu_1, \mu_2 \in \mathcal{I}(q)$. Then $\Phi(\mu_1), \Phi(\mu_2) \in \Phi(\mathcal{I}(q)) = \partial P(q)$. Since $\alpha = \lambda\Phi(\mu_1) + (1-\lambda)\Phi(\mu_2)$ and α is an extreme point, we have

$$\Phi(\mu_1) = \Phi(\mu_2) = \alpha.$$

Hence $\mu_1, \mu_2 \in \mathcal{I}(q,\alpha)$. Since $\mu \in \mathrm{ex}(\mathcal{I}(q,\alpha))$, we have

$$\mu_1 = \mu_2 = \mu.$$

This shows $\mu \in \mathrm{ex}(\mathcal{M}_\sigma(X))$. □

Lemma 4.7. *Let $\alpha \in \partial P(q)$ for some $q > 0$. Then for $\varepsilon > 0$, there exists $\mu \in \mathcal{M}_\sigma(X)$ such that*

$$|\Phi(\mu) - \alpha| < \varepsilon \quad \text{and} \quad |h(\mu) - (P(q) - \alpha q)| < \varepsilon.$$

Proof. We prove in three steps according to different assumptions on α.

Suppose $\alpha = P'(q)$ for some $q > 0$. Then

$$P(q+t) - P(q) = \alpha t + o(|t|). \tag{4.11}$$

By Theorem 3.2, there exists an ergodic measure μ such that

$$P(q) = h(\mu) + q\Phi(\mu) \quad \text{and } P(q+t) \geq h(\mu) + (q+t)\Phi(\mu). \tag{4.12}$$

Hence $P(q+t) - P(q) \geq t\Phi(\mu)$. By (4.11),

$$\alpha t + o(|t|) \geq t\Phi(\mu).$$

Diving by small t with $t > 0$ and $t < 0$ gives

$$\Phi(\mu) = \alpha.$$

This shows $h(\mu) = P(q) - \alpha q$ by (4.12).

Suppose $\alpha \in \partial^e P(q)$. Let $\varepsilon > 0$. By the denseness of the differentiable points of convex function $q \mapsto P(q)$, there exists $t > 0$ such that

$$|P'(t) - \alpha| \leq \varepsilon, \quad |t - q| \leq \varepsilon, \quad |P(t) - P(q)| \leq \varepsilon.$$

By the previous step, there exists an ergodic measure μ such that

$$\Phi(\mu) = P^{'}(t) \quad \text{and} \quad h(\mu) = P(t) - P'(t)t.$$

Together, we have

$$|\Phi(\mu) - \alpha| < \varepsilon \quad \text{and} \quad |h(\mu) - (P(q) - \alpha q)| < \varepsilon.$$

Suppose $\alpha \in \partial P(q) = [\alpha_1, \alpha_2]$. Then

$$\alpha = \lambda\alpha_1 + (1 - \lambda)\alpha_2$$

for some $\lambda \in [0, 1]$. By the previous step, for $i = 1, 2$, there exists an ergodic measure μ_i such that

$$|\Phi(\mu_i) - \alpha_i| < \varepsilon \quad \text{and} \quad |h(\mu_i) - (P(q) - \alpha_i q)| < \varepsilon.$$

Set $\mu := \lambda\mu_i + (1 - \lambda)\mu_{i-1} \in \mathcal{M}_\sigma(X)$. Then the affinity of $\mu \mapsto \Phi(\mu)$ and $\mu \mapsto h(\mu)$ implies

$$|\Phi(\mu) - \alpha| < \varepsilon \quad \text{and} \quad |h(\mu) - (P(q) - \alpha q)| < \varepsilon.$$

This finishes the proof. □

Now we are ready to prove Theorem 4.5.

Proof of Theorem 4.5. We first show (i). By Lemma 4.6, there exists an ergodic measure μ such that

$$P(q) = q\Phi(\mu) + h(q) \quad \text{and} \quad \Phi(\mu) = \alpha.$$

By Kingman's ergodic theorem, $\mu(E(\alpha)) = 1$, thus $E(\alpha) = \emptyset$. Hence

$$h(E(\alpha)) \geq h(\mu) = P(q) - \alpha q = \inf_{t>0}\{P(t) - \alpha t\}$$

where the first inequality is a special case of Theorem 5.2.

Next we move to (ii). By Lemma 4.7, there exist a sequence of measures $(\mu_n) \in \mathcal{M}_\sigma(X)$ such that

$$\lim_{n\to\infty} P(\mu_n) = \alpha \quad \text{and} \quad \limsup h(\mu_n) \geq \inf_{t>0}\{P(t) - \alpha t\}.$$

By passing to a subsequence, we have $\mu_n \xrightarrow{w-*} \mu$ for some $\mu \in \mathcal{M}_\sigma(X)$. The upper semi-continuity of $\mu \mapsto h(\mu)$ and $\mu \mapsto \Phi(\mu)$ implies

$$\Phi(\mu) \geq \alpha \quad \text{and} \quad h(\mu) \geq \inf_{t>0}\{P(t) - \alpha t\} = P(q) - \alpha q.$$

Hence by Theorem 3.2,

$$h(\mu) \geq P(q) - \alpha q \geq P(q) - \Phi(\mu)q \geq h(\mu),$$

which implies

$$\Phi(\mu) = \alpha \quad \text{and} \quad h(\mu) = P(q) - \alpha q.$$

Thus

$$h(\alpha) \geq h(\mu) = \inf_{t>0}\{P(t) - \alpha t\}.$$

This finishes the proof. $\square$

5 Weighted thermodynamic formalism

Let $\pi\colon X \to Y$ be a factor map between subshifts X, Y. Without loss of generality, we can assume that π is an one-block map. In this section, we will study a thermodynamic formalism with respect to the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\sigma_Y} & Y \end{array}$$

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ such that $a_1 + a_2 = 1$, $a_1 > 0$ and $a_2 \geq 0$. Below we introduce the dynamic quantities.

- The $\mathbf{a}$-*weighted measure-theoretic* $h^{\mathbf{a}}\colon \mathcal{M}_\sigma(X) \to [0, \infty)$ is

$$h^{\mathbf{a}}(\mu) := a_1 h(\mu) + a_2 h(\pi\mu) \quad \text{for } \mu \in \mathcal{M}_\sigma(X),$$

  where h denotes the classical measure-theoretic entropy in (2.1).

- Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. The Lyapunov exponent $\Phi\colon \mathcal{M}_\sigma(X) \to [-\infty, \infty)$ is

$$\Phi(\mu) := \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n \, d\mu \quad \text{for } \mu \in \mathcal{M}_\sigma(X).$$

- For $I \in \mathcal{L}_n(X)$, define the corresponding $\mathbf{a}$-*weighted cylinder* as

$$I^{\mathbf{a}} := \Big\{ x \in X \colon x_\ell = I_\ell \text{ for } 1 \leq \ell \leq a_1 n;\ \pi(x_\ell) = \pi(I_\ell) \text{ for } 1 \leq \ell \leq n \Big\}.$$

  Let $\Phi = \{\log \phi_n\}$ be a subadditive potential on X. Let $E \subset X$. For $s \geq 0$ and $n \in \mathbb{N}$, define

$$\mathcal{M}^{\mathbf{a},s}(\Phi, E, n) := \inf \Big\{ \sum_i \exp\big(- s n_i + \frac{1}{a_1} \sup_{x \in I_i^{\mathbf{a}}} \log \phi_{a_1 n_i}(x) \big) \colon \{I_i^{\mathbf{a}}\} \text{ covers } E \text{ with } n_i = |I_i| \geq n \Big\}$$

  and

$$\mathcal{M}^{\mathbf{a},s}(\Phi, E) := \lim_{n\to\infty} \mathcal{M}^{\mathbf{a},s}(\Phi, E, n).$$

Then the $(\mathbf{a},\Phi)$-*weighted topological pressure* $P^{\mathbf{a}}(\Phi,\cdot)\colon 2^X \to [-\infty,\infty)$ is

$$\begin{aligned}P^{\mathbf{a}}(\Phi,E) &:= \inf\{s\geq 0\colon \mathcal{M}^{\mathbf{a},s}(\Phi,E)=0\} = \inf\{s\geq 0\colon \mathcal{M}^{\mathbf{a},s}(\Phi,E)<\infty\}\\ &= \sup\{s\geq 0\colon \mathcal{M}^{\mathbf{a},s}(\Phi,E)>0\} = \sup\{s\geq 0\colon \mathcal{M}^{\mathbf{a},s}(\Phi,E)=\infty\}.\end{aligned}$$

The $\mathbf{a}$-*weighted topological pressure* $P^{\mathbf{a}}\colon O(X)\to[-\infty,\infty)$ is

$$P^{\mathbf{a}}(\Phi) := P^{\mathbf{a}}(\Phi,X).$$

For $\mathbf{a}=(a_1,a_2)$ with $a_1>0, a_2\geq 0$, by convention we write $\widetilde{\mathbf{a}}=\mathbf{a}/\|\mathbf{a}\|_1$ and define

$$h^{\mathbf{a}}(\mu) := \|\mathbf{a}\|_1 h^{\widetilde{\mathbf{a}}}(\mu), \quad P^{\mathbf{a}}(\Phi) := \|\mathbf{a}\|_1 P^{\widetilde{\mathbf{a}}}\left(\frac{\Phi}{\|\mathbf{a}\|_1}\right).$$

5.1 Weighted variational principle

Theorem 5.1 (Weighted variational principle)**.** *Let* $\mathbf{a}=(a_1,a_2)$, $a_1>0, a_2\geq 0$ *and* $\Phi=\{\log\phi_n\}$ *be a subadditive potential on* X*. Then*

$$\max\{\Phi(\mu)+h^{\mathbf{a}}(\mu)\colon \mu\in\mathcal{M}_\sigma(X)\} = P^{\mathbf{a}}(\Phi).$$

Theorem 5.1 is a combination of Theorem 5.2 and Theorem 5.3.

5.1.1 Lower bound

Theorem 5.2. *Let* $\mathbf{a}=(a_1,a_2)$, $a_1>0, a_2\geq 0$ *and* $\Phi=\{\log\phi_n\}$ *be a subadditive potential on* X*. Then for* $\mu\in\mathcal{M}_\sigma(X)$,

$$P^{\mathbf{a}}(\Phi)\geq \Phi(\mu)+h^{\mathbf{a}}(\mu).$$

Theorem 5.2 will be proved following a dynamic mass distribution principle.

Proof. Without loss of generality, we assume $a_1+a_2=1$. First we suppose that μ is ergodic. Then $\pi\mu$ is also ergodic. Write $h_1=h(\mu)$ and $h_2=h(\pi\mu)$. Let $\beta<\Phi(\mu)$. By Shannon-McMillan-Breiman theorem and Kingman's ergodic theorem, there exists $E\subset X$ with $\mu(E)$ close to 1 and $N\in\mathbb{N}$ such that for $n\geq N$ and $x\in E$,

$$\mu(x|n)=\exp(-n(h_1+o(1)),\ (\pi\mu)(\pi(x)|n)=\exp(-n(h_2+o(1))),\ \log\phi_n(x)>n\beta. \quad (5.1)$$

Then for n large and $I\in\mathcal{L}_n(X)$,

$$\#\{J\in\mathcal{L}_n(X)\colon \pi(J)=\pi(I), J\cap E\neq\emptyset\} = \exp(n(h_1-h_2+o(1))).$$

Thus for $I\in\mathcal{L}_n(X)$,

$$\begin{aligned}\mu(I^{\mathbf{a}}\cap E) &= \sum_{J\in\mathcal{L}_n(X)\colon I|a_1n=J|a_1n,\pi(I)=\pi(J),J\cap E\neq\emptyset}\mu(J)\\ &\leq \exp(-nh_1)\#\{J\in\mathcal{L}_{(1-a_1)n}(X)\colon \pi(J)=\pi(I^n_{(1-a_1)n}),\ J\cap E\neq\emptyset\}\\ &= \exp\left(n(-h_1+(1-a_1)(h_1-h_2)+o(1))\right)\\ &= \exp\left(-n(a_1h_1+a_2h_2+o(1))\right)\\ &= \exp\left(-n(h^{\mathbf{a}}(\mu)+o(1))\right)\end{aligned} \quad (5.2)$$

Let $\alpha < h^{\mathbf{a}}(\mu)$. Write $\delta := h^{\mathbf{a}}(\mu) - \alpha > 0$. Let $\{I_i^{\mathbf{a}}\}$ be a cover of E with $|I_i| \geq n$. We can assume $x_i \in I_i^{\mathbf{a}} \cap E$. By taking $o(1) < \delta/2$, we have

$$\begin{aligned}
&\sum_i \exp\big(-n_i(\alpha+\beta) + \frac{1}{a_1}\sup_{x\in I_i^{\mathbf{a}}}\log\phi_{a_1 n_i}(x)\big)\\
&\geq \sum_i \exp\big(-n_i(h^{\mathbf{a}}(\mu)-\delta+\beta) + \frac{1}{a_1}\log\phi_{a_1 n_i}(x_i)\big)\\
&\geq \exp(n\delta/2)\sum_i \exp\big(-n_i(h^{\mathbf{a}}(\mu)+o(1))\big) && \text{by (5.1)}\\
&\geq \exp(n\delta/2)\sum_i \mu(I_i^{\mathbf{a}}\cap E) && \text{by (5.2)}\\
&\geq \mu(E).
\end{aligned}$$

Taking infimum with respect to the **a**-weighted coverings of E and letting $n \to \infty$ shows

$$\mathcal{M}^{\mathbf{a},\alpha+\beta}(\Phi, X) \geq \mathcal{M}^{\mathbf{a},\alpha+\beta}(\Phi, E) \geq \mu(E) \geq \frac{1}{2} > 0.$$

Hence

$$P^{\mathbf{a}}(\Phi) = P^{\mathbf{a}}(\Phi, X) \geq \alpha + \beta.$$

Letting $\alpha \to h^{\mathbf{a}}(\mu)$ and $\beta \to \Phi(\mu)$ gives

$$P^{\mathbf{a}}(\Phi) \geq \Phi(\mu) + h^{\mathbf{a}}(\mu). \tag{5.3}$$

Let $\nu \in \mathcal{M}_\sigma(X)$. The ergodic decomposition gives a probability $\mathbb{P}$ on $\mathrm{ex}(\mathcal{M}_\sigma(X))$ such that

$$\nu = \int \mu \, d\mathbb{P}(\mu).$$

By (5.3), the affinities of $\mu \mapsto \Phi(\mu)$ and $\mu \mapsto h^{\mathbf{a}}(\mu)$ implies that

$$\Phi(\nu) + h^{\mathbf{a}}(\nu) = \int \Phi(\mu) + h^{\mathbf{a}}(\mu)\, d\mathbb{P}(\mu) \leq P^{\mathbf{a}}(\Phi).$$

This completes the proof. □

5.1.2 Upper bound

Theorem 5.3. *Let* $\mathbf{a} = (a_1, a_2)$, $a_1 > 0, a_2 \geq 0$ *and* $\Phi = \{\log\phi_n\}$ *be a subadditive potential on* X*. Then*

$$P^{\mathbf{a}}(\Phi) \leq \sup\{\Phi(\mu) + h^{\mathbf{a}}(\mu) \colon \mu \in \mathcal{M}_\sigma(X)\}$$

Inspired by the classical results about the Hausdorff dimensions of sets and measures, it is natural to ask for a dynamic Frostman lemma. This is the key part in the proof of the upper bound in [11, Theorem 1.4], see [11, Section 5] for details. However, for the symbolic dynamics, we have a direct proof which begins with the Legendre transform of $\mu \mapsto h^{\mathbf{a}}(\mu)$.

Theorem 5.4. *Let* $\mathbf{a} = (a_1, a_2)$ *with* $a_1 > 0, a_2 \geq 0$ *and* $\Phi = \{\log \phi_n\}$ *be a subadditive potential on* X*. Then*

$$\sup\{\Phi(\mu) + h^{\mathbf{a}}(\mu) \colon \mu \in \mathcal{M}_\sigma(X)\} = \|\mathbf{a}\|_1 P(\Psi)$$

where $\Psi = \{\log \psi_n\}$ *is the subadditive potential on* Y *defined by*

$$\psi_n(y) = \left(\sum_{I \in \mathcal{L}_n(X) \colon \pi(I) = y|n} \sup_{x \in I} \phi_n^{1/a_1}(x) \right)^{a_1/(a_1+a_2)} \qquad \text{for } y \in Y,\ n \in \mathbb{N}.$$

Proof. Without loss of generality, we assume $a_1 + a_2 = 1$. The proof is completed by the following optimization process.

$$\begin{aligned}
&\sup_{\mu \in \mathcal{M}_\sigma(X)} \{\Phi(\mu) + h^{\mathbf{a}}(\mu)\} \\
&= \sup_{\nu \in \mathcal{M}_\sigma(Y)} \sup_{\pi\mu = \nu} \{\Phi(\mu) + a_1 h(\mu) + a_2 h(\nu)\} \\
&= \sup_{\nu \in \mathcal{M}_\sigma(Y)} \left\{ a_1 \sup_{\pi\mu = \nu} \left\{ \frac{1}{a_1}\Phi(\mu) + h(\mu) - h(\nu) \right\} + h(\nu) \right\} \\
&= \sup_{\nu \in \mathcal{M}_\sigma(Y)} \{\Psi(\nu) + h(\nu)\} && \text{by Theorem 3.1} \\
&= P(\Psi) && \text{by Theorem 3.2.}
\end{aligned}$$

□

Theorem 5.3 follows from a combination of Theorem 5.4 and the next proposition.

Proposition 5.5. *Let* $\mathbf{a} = (a_1, a_2)$ *with* $a_1 > 0, a_2 \geq 0$ *and* $\Phi = \{\log \phi_n\}$ *be a subadditive potential on* X*. Let* Ψ *be the potential given in Theorem 5.4. Then*

$$P^{\mathbf{a}}(\Phi) \leq \|\mathbf{a}\|_1 P(\Psi).$$

Proof. Without loss of generality, we assume $a_1 + a_2 = 1$. Let $s > P(\Psi)$. Then for n large,

$$\log \left(\sum_{J \in \mathcal{L}_{a_1 n}(Y)} \left(\sum_{I \in \pi^{-1} J} \sup_{x \in I} \phi_{a_1 n}^{1/a_1}(x) \right)^{a_1} \right) < sn. \tag{5.4}$$

For $n \in \mathbb{N}$, define

$$\Gamma_n(X) := \{I^{\mathbf{a}} \colon I \in \mathcal{L}_n(X)\}.$$

Then Γ_n is a cover of X with $\mathbf{a}$-weighted cylinders of length n. Since $[I^{\mathbf{a}}] \subset [I|a_1 n]$,

$$\sup_{x \in I^{\mathbf{a}}} \phi_{a_1 n}(x) \leq \sup_{x \in I|a_1 n} \phi_{a_1 n}(x).$$

Hence

$$\sum_{I^{\mathbf{a}} \in \Gamma_n(X)} \exp\left(\frac{1}{a_1} \sup_{x \in I^{\mathbf{a}}} \log \phi_{a_1 n}(x) \right)$$

$$\begin{aligned}
&\le \sum_{I^{\mathbf{a}} \in \Gamma_n(X)} \sup_{x \in I|a_1 n} \phi_{a_1 n}^{1/a_1}(x) \\
&= \sum_{I \in \mathcal{L}_{a_1 n}(X)} \sup_{x \in I} \phi_{a_1 n}^{1/a_1}(x) \\
&= \sum_{J \in \mathcal{L}_{a_1 n}(Y)} \sum_{I \in \pi^{-1}(J)} \sup_{x \in I} \phi_{a_1 n}^{1/a_1}(x) \\
&= \exp\left(\frac{1}{a_1} \log \left(\sum_{J \in \mathcal{L}_{a_1 n}(Y)} \sum_{I \in \pi^{-1}(J)} \sup_{x \in I} \phi_{a_1 n}^{1/a_1}(x) \right)^{a_1} \right) \\
&\le \exp\left(\frac{1}{a_1} \log \left(\sum_{J \in \mathcal{L}_{a_1 n}(Y)} \left(\sum_{I \in \pi^{-1}(J)} \sup_{x \in I} \phi_{a_1 n}^{1/a_1}(x) \right)^{a_1} \right) \right) \quad \text{by subadditivity of } x \mapsto x^{a_1} \\
&\lesssim \exp(sn) \qquad \text{by (5.4).}
\end{aligned}$$

This implies

$$\mathcal{M}^{\mathbf{a},s}(\Phi, X, n) \le \exp(-sn) \sum_{I^{\mathbf{a}} \in \Gamma_n(X)} \exp\left(\frac{1}{a_1} \sup_{x \in I^{\mathbf{a}}} \log \phi_{a_1 n}(x) \right) \lesssim 1.$$

Letting $n \to \infty$ gives $\mathcal{M}^{\mathbf{a},s}(\Phi, X) \lesssim 1$. Thus

$$P^{\mathbf{a}}(\Phi) \le s.$$

Letting $s \to P(\Psi)$ gives $P^{\mathbf{a}}(\Phi) \le P(\Psi)$. □

5.2 Unique weighted equilibrium state

Recall the definitions of weak specification, $\Theta(X)$ and $\Omega(X)$ from subsection 3.2.

Theorem 5.6 (Unique weighted equilibrium state). *Let $\pi\colon X \to Y$ be an one-block factor map between subshifts X, Y. Suppose X satisfies weak specification. Let $\phi \in \Theta(X)$ and $\Phi = \{\log \phi_n\}$ be the subadditive potential induced from ϕ. Let $\mathbf{a} = (a_1, a_2)$, $a_1 > 0, a_2 \ge 0$. Then there is a unique $\mathbf{a}$-weighted equilibrium state μ, that is,*

$$\Phi(\mu) + h^{\mathbf{a}}(\mu) = P^{\mathbf{a}}(\Phi).$$

Moreover, μ is ergodic and for $I \in \mathcal{L}(X)$,

$$\mu(I) \gtrsim \widetilde{\phi}(I)$$

where

$$\widetilde{\phi}(I) := \frac{\phi(I)^{1/a_1}}{\psi(\pi(I))^{(a_1+a_2)/a_1}} \frac{\psi(\pi(I))}{Z_{|I|}}$$

in which,

$$\psi(J) := \left(\sum_{\pi(I)=J} \phi^{1/a_1}(I) \right)^{a_1/(a_1+a_2)} \qquad \text{for } J \in \mathcal{L}(Y)$$

and

$$Z_n := \sum_{J \in \mathcal{L}_n(Y)} \psi(J) \qquad \text{for } n \in \mathbb{N}.$$

Proof. Without loss of generality we assume $a_1 + a_2 = 1$. Define

$$f(I) := \psi(\pi(I))^{a_2/a_1} Z_{|I|} \quad \text{for } I \in \mathcal{L}(X).$$

Since $\psi \circ \pi$ is subadditive and $Z_n \approx \exp(nP(\Psi))$, it is readily checked that

$$\widetilde{\phi} = \frac{\phi^{1/a_1}}{f} \quad \in \Omega(X).$$

By Proposition 3.5, there is an ergodic measure μ such that

$$\mu(I) \gtrsim \widetilde{\phi}(I) \approx \frac{\phi(I)^{1/a_1}}{\psi(\pi(I))^{1/a_1 - 1} \exp(nP(\Psi))} \quad \text{for } I \in \mathcal{L}(X). \tag{5.5}$$

Let η be any **a**-weighted equilibrium state provided by Theorem 5.4. Using Theorem 3.1 and Theorem 3.4, we conclude from the proof of Theorem 5.4 that

$$\pi\eta = \nu \tag{5.6}$$

and

$$\frac{1}{a_1}\Phi(\eta) + h(\eta) = \frac{1}{a_1}\Psi(\nu) + h(\nu) \tag{5.7}$$

where ν satisfies

$$\Psi(\nu) + h(\nu) = P(\Psi) \tag{5.8}$$

and

$$\sum_{I \in \mathcal{L}_n(Y)} -\nu(J) \log \nu(J) = nh(\nu) + O(1), \quad \sum_{J \in \mathcal{L}_n(Y)} -\nu(J) \log \psi(J) = n\Psi(\nu) + O(1). \tag{5.9}$$

Finally,

$$\begin{aligned}
&\sum_{I \in \mathcal{L}_n(X)} -\eta(I) \log \eta(I) + \eta(I) \log \mu(I) \\
&\geq \sum_{I \in \mathcal{L}_n(X)} -\eta(I) \log \eta(I) + \eta(I) \log \frac{\phi(I)^{1/a_1}}{\psi(\pi(I))^{1/a_1 - 1} \exp(nP(\Psi))} + O(1) && \text{by (5.5)} \\
&\geq \sum_{I \in \mathcal{L}_n(X)} -\eta(I) \log \eta(I) + \frac{1}{a_1}\eta(I) \log \phi(I) + \\
&\qquad - \sum_{J \in \mathcal{L}_n(Y)} (\frac{1}{a_1} - 1)\nu(I) \log \psi(J) - nP(\Psi) + O(1) && \text{by (5.6)} \\
&\geq nh(\eta) + \frac{n}{a_1}\Phi(\eta) - n(\frac{1}{a_1} - 1)\Psi(\nu) - nP(\Psi) + O(1) && \text{by (5.9)} \\
&= nh(\eta) + \frac{n}{a_1}\Phi(\eta) - \frac{n}{a_1}\Psi(\nu) - nh(\nu) + O(1) && \text{by (5.8)} \\
&= O(1) && \text{by (5.7).}
\end{aligned}$$

This shows $\eta \ll \mu$ by Lemma 2.12. Thus $\eta = \mu$ since μ is ergodic. □

6 Further questions

Here are some questions based on the guiding principle (1.1). We thank Professor Feng for some insightful discussions about these problems. Moreover, he has recommended some references and encouraged us to the study of specific examples before the abstract generalization.

1. Can we endow the set of subadditive potentials $O(X)$ with some natural topology (or infinite manifold structure) such that we can conduct the convex analysis with respect to $O(X)$ and $\mathcal{M}_\sigma(X)$ just like the classical thermodynamic formalism?

   Note that $O(X)$ is a subset of infinite product space. The measures on symbolic space can also be characterized by the discretizations along the tree of cylinders.

   The suggested reference for this question is [8].

2. How about combining Section 5 and Section 4? This means that can we analyze the multifractal spectrum constructed from the pointwise Lyapunov exponent

$$\Phi\colon X \to [-\infty, \infty]$$

   and **a**-weighted topological pressure

$$P^{\mathbf{a}}(\Phi, \cdot)\colon 2^X \to \mathbb{R}$$

   using the weighted variational principle

$$\sup_{\mu \in \mathcal{M}_\sigma(X)} \{\Phi(\mu) + h^{\mathbf{a}}(\mu)\} = P^{\mathbf{a}}(\Phi) \quad ?$$

   We are told that there is some related work in [1].

3. Can we analyze a more general multifractal spectrum based on the set-valued Lyapunov exponent

$$\Phi\colon X \to 2^{\mathbb{R}}$$

   defined by

$$\Phi(x) := \text{ the limit points of } \left\{\frac{1}{n} \log \phi_n(x)\right\}_{n=1}^{\infty} \quad ?$$

   This is inspired by [3, Exercise 1.48].

   It is suggested that some ideas can be found in Lars Olsen's work about the *divergence points*.

4. How about the singular value potential? What if we do the multifractal analysis for the Lyapunov exponent defined by this one-parameter family of potential?

   The recommended reference is [14].

A Draft reflections

- In the dimension theory of iterated function system and smooth dynamics, the subadditive thermodynamical formalism are more useful than the classical (additive) ones since the potentials are usually subadditive or supdadditive.
- Thermodynamic formalism and mutifractal formalism are nothing but the convex analysis and optimization in infinite dimensional space. The corresponding dual pair of 'spaces' are the set of (subadditive) potentials and the set of invariant measures.
- Thermodynamic formalism is usually about the first convex conjugate while multifractal formalism tends to focus on the double conjugates and establish some Fenchel duality and variational principle betweens sets and measures.
- The variational principle in thermodynamic formalism says that the topological pressure is the Legendre transform of the negative measure entropy.
- The founding fathers of the convex analysis: Hermann Minkowski; Werner Fenchel; Jean-Jacques Moreau; Rockafellar.
- Why not compare the self-adjoint extension of unbounded operators by double conjugates in quantum mechanics to the closure extension of convex functions by double convex conjugates in convex analysis?
- If the potential is defined by some measure on balls, then the multifractal analysis for Lyapunov expoents with respect to that potential becomes the multifractal analysis of the local dimensions of that measure.
- "Ruelle's 'Thermodynamic Formalism' is the book with the deepest insight in the field of thermodynamics." – Professor Feng.
- There are $3 \times 2 = 6$ canonical mutifractal spectra which are {Dimension, Entropy, Lyapunov exponent} $\times$ {Dimension, Entropy}. They are called (A, B)-multifractal spectrum where A is a point function and B is a set function. However, there some other spectra related in the dimension theory. For example, the Renyi spectrum (information spectrum or L^q-spectrum), and some other interpolating dimension spectrum (including the intermediate dimension spectrum and Assaud spectrum).
- Roughly, the *multfractal rigidity* says that for a dynamic system,
  - The equivalence of (D, D) and (E, E) spectrum will imply the equivalence of other spectra.
  - A combination of topological equivalence and some multifractal equivalence will imply the smooth equivalence.
- For circle homeomorphisms, no periodic point will imply the *unique ergodicity*. On the other hand, the specification, which in some sense means the denseness of periodic points, assures the uniqueness of equillibrium state, so called the *instrinstic ergodicity*.

- Is there a concept called *upper weak specification* which generalizes the subadditivity?
- All the topological pressures presented in standard ergodic theory textbook are defined in box-counting way. The box-counting like topological pressures are good enough for the dimension theory in conformal setting. However, in nonconformal or affine settings where the Lyapunov spectrum is nontrivial (or the Oseledets splitting is not trivial), it will be more reasonable to consider the Hausdorff-like topological pressure.
- It seems that the measure-theoretic entropy is also defined in a box-counting way? Can we define the measure-theoretic entropy in a Hausdorff way? After a further thought, the process of taking supremum with respect to finite partitions has already 'relaxed' the 'box-counting partition' to 'Hausdorff partition'. Possibly we do not have to worry about this.
- The Riesz representation theorem transfers the measure construction into the construction of continuous functionals where the Hahn-Banach theorem plays the role of 'nuclear weapon'. The Howroyd's construction of Frostman measure can be viewed as an example of the Hahn-Banach construction.
- We should not view the energy as a double integral but an iterated integral. In such way, the energy becomes the average of potentials.
- From some perspectives, essentially we are always solving some optimization problems. The differences from the normal optimization problem is that our variables are measures and coverings.
- In Euclidean space (or some other space with grid), we can attach the shrinking grid dynamics to every geometric sets. However, the problem is that the scenery we explore with respect to different zoom-in ratios can be dramatically different. This is related to Furstenberg's concept of dynamic disjointness.
- There are some advantages of subshifts over the general TDS.
  - The alphabet is finite. Hence the topological entropy is finite.
  - The shift map is expansive. Hence the measure-theoretic entropy is always upper semi-continuous.
  - The canonical metric is an ultra metric. Hence the spanning sets, separating sets, and Bowen balls become simply the cylinders. This leads to the relatively easier definitions of topological entropy and pressures, especially avoiding the use of discretization constant.
  - There is a natural strong generator for the Borel σ-algebra . Then we can reduce the measure-theoretic entropy the dynamic partition entropy with respect to the generator.
  - A combination of last two items avoids some complicated works in connecting partitions, separating sets and coverings by Bowen balls.

– The net (or tree) structure provides a convenient way to reduce the overlapping of coverings, which behaves better than the Besicovitch or Vitali covering theorems in Euclidean space.

B A growing list of history in keywords

- Thermodynamics → Statistical Mechanics → Dynamic system and Ergodic Theory.
- Carnot (18th c.): The father of thermodynamics; Carnot cycle.
- Clausius (19th c.): One of the central founding fathers of the science of thermodynamics; (1850) first stated the basic ideas of the second law of thermodynamics; (1865) Gave the first mathematical version of the concept of entropy, and also gave it its name.
- Maxwell (19th c.): Maxwell's equations; Maxwell–Boltzmann distribution...
- Boltzmann (19th c.): Boltzmann {entropy, constant, distribution, equation}, H-theorem.
- Gibbs (19th c.): Boltzmann distributions are also called Gibbs distributions; Gibbs' inequality; Gibbs' H-theorem; Gibbs entropy formula; Together with James Clerk Maxwell and Ludwig Boltzmann, Gibbs founded *statistical mechanics*; Develop the formalism of equilibrium statistical mechanics on finite phase spaces – which we shall call *thermodynamic formalism*.
- Von Neumann 1932: Using the viewpoint of Koopman (so-called Koopman operator) and apply operator theory to obtain the L^2 ergodic therem.
- Birkhoff 1931: Pointwise Ergodic Theorem.
- Shannon 1948: The entropy in information theory (called *Shannon entropy*) and establish some properties.
- Kolmogorov-Sinai 1958: Define the measure-theoretic entropy
- Adler-Konheim-McAndrew 1965: Define the topological entropy (for compact sets)
- Sinai: Introduce the Markov partition and symbolic dynamics for Anosov diffeomorphism
- Furstenberg-Kesten 1960: the study of products of random matrices
- Rokhlin 1962: The theory of conditional measures
- Oseledets 1968: Multiplicative Ergodic Theorem (MET).
- Dinaburg, Goodman, Goodwyn 1969–1971: establish the variational principle for the topological entropy.

- Ruelle:
  - (1973) Define topological pressure. Prove the variational principle and establish the existence of equilibrium state in the setup of symbolic dynamics
  - Apply the tool of Ruelle-Perron-Frobenius Operator (transfer operator).
  - (1982): MET for compact operator cocycle on Hilbert space.
- Walters 1975: Extend Ruelle's work of variational principle to continuous potential on compact metric space.
- Parry 1964: Obtain the thermodynamics for sofic systems which includes:
  - Variational principle.
  - Existence and uniqueness of equilibrium state (called *Parry measure*).
  - The Gibbs property of the equilibrium state.
- Raghunathan 1979: Prove MET in a way different from Oseledec.
- Smale:
  - Poincaré conjecture in dimension ≥ 5
  - Smale's horseshoe map: a classic motivating example in hyperbolic dynamics
  - The concept of Axiom A diffeomorphism
- Bowen:
  - (1973) Define the topological entropy for *non-compact* sets.
  - (1975) Prove the uniqueness of equilibrium state under the condition of expansive map and specification.
  - (1979) Bowen's equation for Hausdorff dimension of conformal attractors or repellers as the zero of a pressure function
  - Establish the Markov partition for Axiom A diffeomorphism using *shadowing*
- Ornstein 1970: the complete classification of Bernoulli shift with entropy
- Pesin:
  - (1976) Pesin theory for invariant manifolds: The construction of stable, center, and unstable submanifolds in smooth dynamics
  - Pesin's formula for entropy, (cf. Ruelle-Margulis inequality)
- Mane 1981: a version of MET for compact operators on Banach space under some continuity assumptions on the base dynamics and the dependence of the operator on the base point.
- Misiurewicz 1976: A short (elegant) proof of the variational principle for $\mathbb{Z}^d$ action on compact space

- Margulis: homogeneous dynamics.
- Furstenberg
  - The tool of scenary flow
  - Introduce the concept of disjointness of dynamics and give some conjectures
  - The concept of *dimension conservation* in the relation of projected dimension and fiber dimensions
- Froyland-Lloyd-Quas 2010: Coherent structures and isolated spectrum for Perron-Frobenius cocycles (used in [Feng 2019]).

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