Intermediate dims. under self-affine codings

Oulu Analysis Seminar, 12/05/2023 (Zhou Feng).

§ Intro

§§ setup

- Let $T_1, \cdots, T_m \in \mathbb{R}^{d\times d}$, $\|T_j\| < 1$

- For $a = (a_1, \cdots, a_m) \in \mathbb{R}^{d\times m}$, consider the IFS

$$\{ f_j^a (x) = T_j x + a_j \}_{j=1}^m .$$

whose attractor denoted by K^a.

- The coding map $\pi^a : \Sigma = \{1, \cdots, m\}^{\mathbb{N}} \to \mathbb{R}^d$ is

$$\pi^a(\mathbf{i}) := \lim_{n\to\infty} f_{i_1} \circ \cdots f_{i_n}(0) \quad \text{for } \mathbf{i} = (i_k) \in \Sigma.$$

- Theme: Study various dimensional properties of the projected sets and measures under π^a.

e.g. $\begin{cases} \text{"exact-dim"} \\ \text{constancy result} \\ \text{estimate the exceptional set.} \end{cases}$

§§ Progress

- (Falconer 1988) If $\|T_j\| < \frac{1}{3}$ for all j, then

$$\dim_H K^a = \dim_B K^a = \min\{\dim_{AFF}\Sigma, d\}$$

where $\dim_{AFF}\Sigma$ is the zero of the pressure w.r.t. the singular value potential, called affinity dim. Note that $\dim_{AFF}\Sigma$ has a potential-theoretical characterization:

$$\dim_{AFF}\Sigma = \sup\left\{ s \geq 0 : \exists\, \mu \in P(\Sigma) \iint \frac{1}{\phi^s(x \wedge y)}\, d\mu(x)\, d\mu(y) < \infty \right\}.$$

where $\phi^s(x \wedge y) := \phi^s(T_{x \wedge y})$.

- (Solomyak 1988) $\|T_j\| < \frac{1}{3} \xrightarrow{\text{relaxed}} \|T_j\| < \frac{1}{2}$.

Assume $\|T_j\| < \frac{1}{2}$. (Mention some results)

- (Käenmäki 2004) $\exists$ ergodic $\mu \in \mathcal{E}_\sigma(\Sigma)$ s.t.

$$\underline{\dim}_H \pi^a \mu = \dim_H \pi^a(\Sigma) \text{ for } \mathcal{L}\text{-a.e. } a.$$

- (Jordan-Pollicott-Simon 2007) $\forall \mu \in \mathcal{E}_\sigma(\Sigma)$,

$$\dim_H \pi^a \mu = \dim_{LY} \mu \text{ for } \mathcal{L}\text{-a.e. } a.$$

- (Käenmäki-Vilppolainen 2010) For $E \subset \Sigma$ with

$\sigma E \subset E$,

$$\dim_H \pi^a(E) = \dim_B \pi^a(E) = (\text{zero of some pressure})$$

- (Järvenpää's, Kaenmaki, Koivusalo, Stenflo, Suomala)

- (Järvenpää's, Wu, Wen Wu 2017)

Random affine code tree fractals.

• (Feng - Lo - Ma 2022). A systematic study of various dim. properties of the projected Borel sets and measures. In particular, for Borel $E \subset \Sigma$, each of the Haus. packing, lower, and upper box. dim of $\pi^a(E)$ is constant for λ-a.e. a.

Q: How about the intermediate dims.?

to be recalled later.

A: An anagolous constancy result holds for intermediate dims. Moreover, we extend the results to the generalized intermediate dims in several settings.

$$\left\{\begin{array}{l}\text{self-affine codings}\\ \text{othogonal projections}\\ \text{images of fractional B.M.}\end{array}\right.$$

§3 Main Results

Recall the upper θ-int. dim. introduced by Falconer, Fraser, Kempton (2020, Math. Z.), for $E \subset \mathbb{R}^d$,

$$\overline{\dim}_\theta E := \inf \Big\{ s \geq 0 : \ \forall \varepsilon > 0,\ \exists r_0 > 0.\ \forall r \in (0, r_0).$$
$$\text{s.t. } \exists \{U_i\} \text{ cover } E \text{ with}$$
$$\sum_i |U_i|^s \leq \varepsilon \text{ and } r^{1/\theta} \leq |U_i| \leq r \Big\}.$$

Then $\quad \dim_H E = \overline{\dim}_0 E \leq \overline{\dim}_1 E \leq \overline{\dim}_B E.$

For simplicity, we focus on $\overline{\dim}_\theta$

Similar results and arguments work for $\underline{\dim}_\theta$.

Many applications of $\overline{\dim}_\theta$ rely on the continuity of $\theta \mapsto \overline{\dim}_\theta E$ at $\theta = 0$.

Thm A

(UB) For all $a \in \mathbb{R}^{dm}$,

$$\overline{\dim}_\theta \pi^a(E) \leq \overline{\dim}_{C,\theta} E.$$

(LB) Assume $\|T_j\| < \frac{1}{2}$ for all j. Then for $\mathcal{L}$-a.e. a,

$$\overline{\dim}_\theta \pi^a(E) = \overline{\dim}_{C,\theta} E.$$

The capacity dimensions will be introduced later.

By replacing the size condition $r^{1/\theta} \leq |U_i| \leq r$ with $\Phi(r) \leq |U_i| \leq r$, Banaj (2021) generalized the θ-int. dim. to so-called Φ-int. dim.

<u>Thm B</u> Let Φ be an admissible function.

Suppose $\lim_{r\to 0} r^{\varepsilon} \log \Phi(r) = 0$ for all $\varepsilon > 0$.

(i) Let $E \subset \Sigma$. Then

$$\begin{cases} \text{For all } a \in \mathbb{R}^{dm}, & \overline{\dim}_{\Phi} \pi^{a}(E) \le \overline{\dim}_{C,\Phi} E. \\ \text{For } \lambda\text{-a.e. } a \in \mathbb{R}^{d\cdot m}, & \overline{\dim}_{\Phi} \pi^{a}(E) = \overline{\dim}_{C,\Phi} E. \end{cases}$$

(ii) Let $E \subset \mathbb{R}^d$. Then

$$\begin{cases} \text{For all } V \in G(d,m), & \overline{\dim}_{\Phi} P_V E \le \overline{\dim}^{m}_{\Phi} E \\ \text{For } \gamma_{d,m}\text{-a.e. } V \in G(d,m), & \overline{\dim}_{\Phi} P_V E = \overline{\dim}^{m}_{\Phi} E. \end{cases}$$

(iii) Let $B_\alpha : \mathbb{R}^d \to \mathbb{R}^m$ be a index-α fractional Brownian motion $(0<\alpha<1)$.

Let $E \subset \mathbb{R}^d$. Then almost surely,

$$\overline{\dim}_{\Phi} B_\alpha(E) = \frac{1}{\alpha} \overline{\dim}^{\alpha m}_{\Phi_\alpha} E.$$

Thm A and Thm B are proved through a capacity approach by adapting and extending some ideas in ([Falconer 2021 "A capacity appr. to box"] [Burrel Falconer Fraser 2021. "Proj. Thms. 4 int. dim."] [Feng-Lo-Ma 2022 "Dims. of proj. sets & mea. on typical self-affine sets"]).

We remark the our kernels are inspired by, but different from that of [B.-F.-F. 2021].
It is these new kernels the reveal a unified computational scheme and pave the way for the extensions to Φ-Int. dims.

For simplicity and clearity, we focus on the proof of Thm A.

§ Preparations

§§ θ-int. dims.

Following [B-F-F], we work on the more convenient "cover sums".

Let $s \geq 0$. For $E \subset \mathbb{R}^d$,

$$S^s_{\theta,r}(E) := \inf \left\{ \sum_i |U_i|^s : \{U_i\} \text{ covers } E \text{ and } r^{1/\theta} \leq |U_i| \leq r \right\}$$

Then

readily checked by def. of $\overline{\dim}_\theta$

$$\overline{\dim}_\theta E = \inf \{ s \geq 0 : \overline{\lim_{r \to 0}}\, S^s_{\theta,r}(E) = 0 \}$$

$$= \left(\text{unique } s \in [0,d] \text{ s.t. } \overline{\lim_{r\to 0}} \frac{\log S^s_{\theta,r}(E)}{\log r} = 0 \right).$$

relies on specialty of $r \mapsto r^{1/\theta}$ $\Leftarrow$ $\underline{\lim}_{r\to 0} \frac{\log r}{\log \Phi(r)} > 0$.

§§ θ-cap. dims.

Concepts in potential theory:

$$\underset{\text{kernel}}{K} \quad \overset{\text{potential}}{P} \quad \underset{\text{energy}}{E} \quad \overset{\text{capacity}}{C}$$

• Let $0<r\leq 1$. The "counting kernel" is

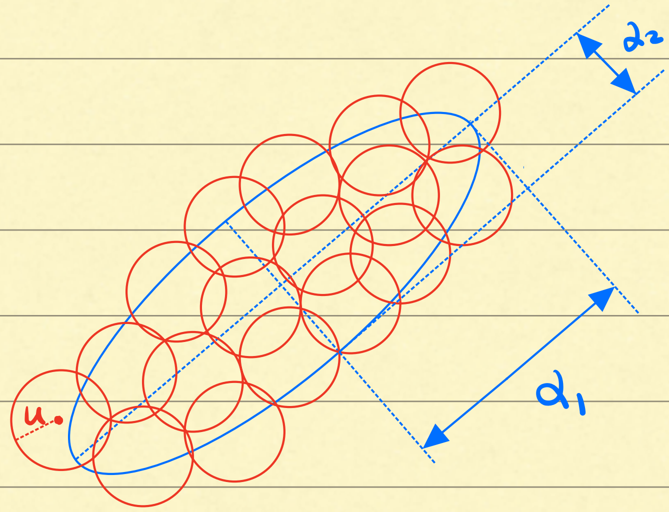
$$Z_r(x\wedge y) := \begin{cases} \prod_{k=1}^{d} \min\left\{1, \dfrac{r}{\alpha_k(T_{x\wedge y})}\right\} & x\neq y \\ 1 & x=y. \end{cases}$$

for $x, y \in \Sigma$.

where $\alpha_1(T_{x\wedge y}) \geq \cdots \geq \alpha_d(T_{x\wedge y})$ are the singular values of $T_{x\wedge y} := T_{(x\wedge y)_1} \cdots T_{(x\wedge y)_d}$.

The geometric intuition of $Z_r(x\wedge y)$ is that

$$N_r(\pi([I])) \lesssim \frac{1}{Z_r(I)}.$$



It relates to the goal of replacing an element $\pi(\ell[I])$ in a cover with $\frac{1}{Z_u(I)}$ many sets of appropriate diameter $u \in [r^{1/\theta}, r]$.

- Let $s \geq 0$. The (desired) kernel is

$$J^s_{\theta,r}(x \wedge y) := \max_{r^{1/\theta} \leq u \leq r} u^{-s} Z_u(x \wedge y).$$

- For compact $E \subset \Sigma$, the capacity is

$$C^s_{\theta,r}(E) := \left(\inf_{\mu \in \mathcal{P}(E)} \iint J^s_{\theta,r}(x \wedge y)\, d\mu(x)\, d\mu(y) \right)^{-1}$$

By convention, $C^s_{\theta,r}(E) := C^s_{\theta,r}(\overline{E})$ for non-cpt. $E \subset \Sigma$.

• Define the lower θ-cap. dim of $E \subset \Sigma$ by

$$\overline{\dim}_{\theta,C} E := \inf \left\{ s \geq 0 : \overline{\lim_{r\to 0}}\, C^s_{\theta,r}(E) = 0 \right\}$$

specialty of $r \mapsto r^{1/\theta}$

$$= \left(\text{unique } s \in [0,d] \text{ s.t. } \overline{\lim_{r\to 0}} \frac{\log C^s_{\theta,r}(E)}{\log r} = 0 \right)$$

By the above argument, the problem reduces to the study of the relationship between

$S^s_{\theta,r}(E)$ and $C^s_{\theta,r}(E)$.

The major tool is the "potential-theoretical version" of the classical "mass distribution principle", which is implicity contained in [Falconer 2021].

§ Proof of (UB)

Prop Let $E \subset \Sigma$. If $\exists\ \mu \in \mathcal{P}(E)$ and $\gamma > 0$

s.t.
$$\int J^{s}_{\theta,r}(x \wedge y)\, d\mu(y) \geq \gamma \quad \text{for all } x \in E.$$

then for all sufficently small $r > 0$,
$$S^{s}_{\theta,r}(\pi^{a}(E)) \lesssim_{d,a,\theta} \frac{\log(1/r)}{\gamma}.$$

Analogy with the (UB) part of m.d.p.:

Let $E \subset \mathbb{R}^d$. If $\exists\ \mu \in \mathcal{P}(E)$ s.t.
$$\frac{\mu(B(x,u))}{u^s} \geq \gamma \qquad \text{for } \forall x \in E,\ \forall u \leq r$$

then
$$\mathcal{H}^{s}_{r}(E) \lesssim \frac{1}{\gamma}.$$

Strategy of the Pf:

- By a discretization, at each $x \in E$, we can pick a "ball" (cylinder) with large "density" up to a constant $\approx \frac{1}{\theta} \log(1/r)$.
- Reduce the overlapping and Project down the selected cylinders to $\mathbb{R}^d$. We obtain a cover of $\pi^a(E)$.
- Replace each $\pi^a([I])$ with a collection of sets of a common appropriate diameter in $[r^{1/\theta}, r]$.
- Do the estimates.

Pf: Prove it if time permits.

$\square$

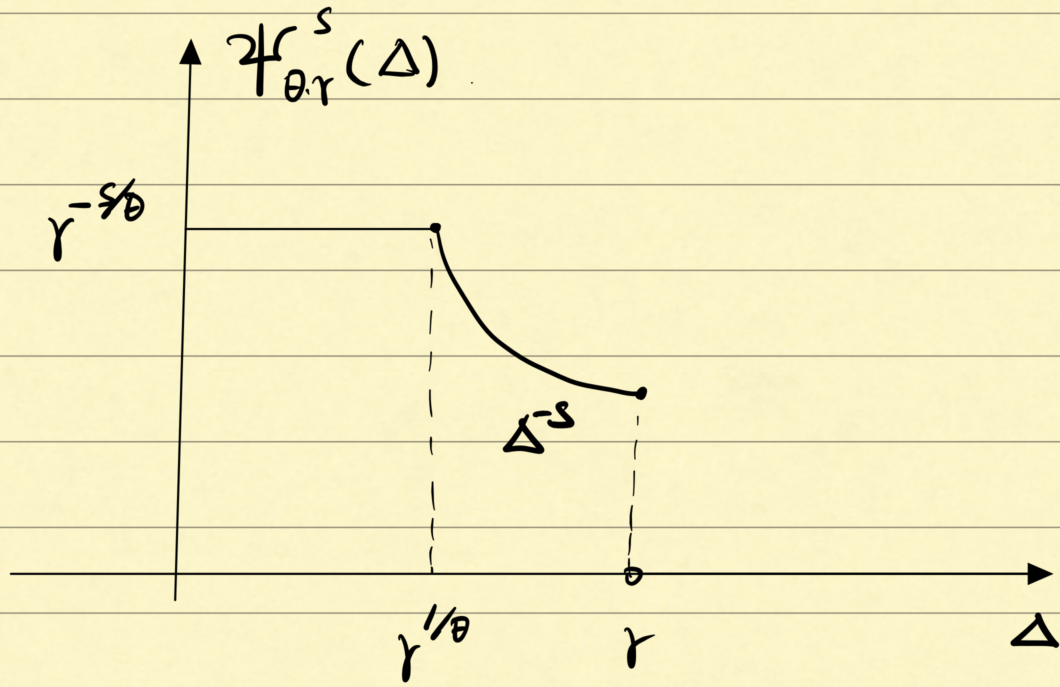
§ Proof of (LB)

Define the "truncated" θ-int. dim. kernel

$$\mathcal{K}^s_{\theta,r}(\Delta) := \begin{cases} r^{-s/\theta}, & 0 \le \Delta \le r^{1/\theta} \\ \Delta^{-s}, & r^{1/\theta} < \Delta \le r \\ 0, & \Delta > r \end{cases} \qquad \Delta \geq 0.$$

$|x-y|$

with graph.



Lem: Let $E \subset \mathbb{R}^d$. If $\exists\, \mu \in \mathcal{P}(E)$ and $F \subset E$, and $\gamma > 0$ s.t.

$$\int \mathcal{H}^s_{\theta,r}(|x-y|)\, d\mu(y) \leq \gamma \quad \text{for all } x \in F,$$

then

$$S^s_{\theta,r}(E) \geq \frac{\mu(F)}{\gamma}.$$

An analogy with the (LB) part of m.d.p.:

Let $E \subset \mathbb{R}^d$. If $\exists\, \mu \in \mathcal{P}(E)$ and $F \subset E$, and γ s.t.

$$\frac{\mu(B(x,u))}{u^s} \leq \gamma \quad \text{for } 0 < u \leq r.$$

then $\mathcal{H}^s_r(E) \gtrsim \dfrac{\mu(F)}{\gamma}$.

We omit the proof since it follows directly from the def. of $\mathcal{H}^s_{\theta,r}(\cdot)$.

To establish the almost all result, an app. of Fubini like for $\mu \in P(E)$

$$\int_{B_\rho} \iint_{\pi^a(E) \times \pi^a(E)} \psi^s_{\theta, r}(|u-v|) \, d\pi^a\mu(u) \, d\pi^a\mu(v) \, da$$

$$= \iint_{E \times E} \left(\underbrace{\int_{B_\rho} \psi^s_{\theta r}(|\pi^a(x) - \pi^a(y)|) \, da}_{} \right) d\mu(x) \, d\mu(y)$$

leads us to control , where the concept of transversality comes in.

<u>Lem</u> (self-affine transversality). see e.g. [J-P-S 2007 CMP]

Assume $\|T_j\| < \frac{1}{2}$. Then

$$\mathcal{L}\{a \in B_\rho : |\pi^a(x) - \pi^a(y)| \le r\} \lesssim_\rho Z_r(x \wedge y).$$

Prop (ParaInt)

$$\int_{B_\rho} \Psi^{s}_{\theta,r}\left(|\pi^a(x)-\pi^a(y)|\right)\, da \lesssim_{\rho,d,\theta} \log(1/r)\, J^{s}_{\theta,r}(x\wedge y).$$

Pf: Show the unified computation scheme if time permits

$\square$

Pf of (LB):

It suffices to prove

$$\overline{\dim}_\theta \pi^a(E) \geq \overline{\dim}_{C.\theta} E.$$

for $\mathcal{L}$-a.e. $a \in B_\rho$ and $\rho > 0$.

Let $0 \leq s \leq d$.

Take a seq. $(r_k) \downarrow 0$ with $0 < r_k \lesssim 2^{-k}$ s.t.

$$\lim_{k\to\infty} \frac{\log C^s_{\theta, r_k}(E)}{-\log r_k} = \overline{\lim_{r\to 0}} \frac{\log C^s_{\theta, r}(E)}{-\log r}$$

For each $k \in \mathbb{N}$, by classical potential theory about the equilibrium measure for cts. positive symmetric kernels,

$\exists\, \mu_k \in \mathcal{P}(E)$, s.t.

$$\iint J^s_{\theta, r_k}(x \wedge y)\, d\mu_k(x)\, d\mu_k(y) = \frac{1}{C^s_{\theta, r_k}(E)} = \gamma_k.$$

By (ParaInt)

$$\iint \int_{B_\rho} \psi^s_{\theta, r_k}(|\pi^a(x) - \pi^a(y)|)\, da\, d\mu_k(x)\, d\mu_k(y)$$

$$\lesssim \log(1/r_k)\, \gamma_k$$

Let $\varepsilon > 0$. By summing over $k \in \mathbb{N}$ and Fubini's.

$$\int_{B_\rho} \sum_k r_k^\varepsilon \gamma_k^{-1} \iint \psi^s_{\theta, r_k}(|\pi^a(x) - \pi^a(y)|)\, d\mu_k(x)\, d\mu_k(y)\, da$$

$$\lesssim \sum_k r_k^\varepsilon \log(1/r_k) \leq A \sum_k r_k^{\frac{\varepsilon}{2}} < \infty$$

Hence $\mathcal{L}$-a.e. a, $\exists\, M_a > 0$. s.t.

$$\iint \psi^s_{\theta, r_k}(|\pi^a(x) - \pi^a(y)|)\, d\mu_k(x)\, d\mu_k(y) \leq M_a \gamma_k \cdot r_k^{-\varepsilon}$$

$\exists\ F_k \subset \pi^a(E)$, with $\pi^a \mu_k(F_k) \geq \frac{1}{2}$ s.t.

$\forall\ u \in F_k$,

$$\int k^s_{\theta, r_k}(|u-v|)\, d\pi^a \mu(v) \leq 2 M_a r_k \cdot r_k^{-\varepsilon}.$$

By Potential-version M.D.P.

$$S^s_{\theta, r_k}(\pi^a(E)) \geq \frac{1}{2} \cdot \frac{r_k^{\varepsilon}}{2 M_a r_k} \gtrsim_a r_k^{\varepsilon}\, C^s_{\theta, r_k}(E).$$

The proof is finished by taking

$\log$, $\lim_{k \to \infty}$, $\lim_{\varepsilon \to 0}$.

$\square$