

On dimension rigidity of planar self-conformal measures

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Where? Setup and examples

We study the dimensions of the self-conformal set and measures associated to a **holomorphic IFS** $\Phi = \{\varphi_i\}_{i \in \Lambda}$ on a bounded domain $\Omega \subset \mathbb{C}$. That is, $\#\Lambda < \infty$, and each φ_i is a holomorphic injection on a domain $\Omega' \supset \bar{\Omega}$, and $\varphi_i(\bar{\Omega}) \subset \Omega$ with $0 < |\varphi'_i(z)| < 1$ for all $z \in \bar{\Omega}$. The **self-conformal set** K is the unique nonempty compact subset of $\mathbb{C}$ such that $K = \bigcup_{i \in \Lambda} \varphi_i(K)$. Given a positive probability vector $p = (p_i)_{i \in \Lambda}$, the **self-conformal measure** μ is the unique Borel probability measure on $\mathbb{C}$ such that $\mu = \sum_{i \in \Lambda} p_i \cdot \varphi_{i\#} \mu$.

Dimension of μ

$$\dim \mu = \lim_{r \rightarrow 0} \frac{\log \mu(B(z, r))}{\log r} \quad \text{for } \mu\text{-a.e. } z$$

(D.-J. Feng–H. Hu ('09))

Entropy of p

$$H(p) = \sum_{i \in \Lambda} -p_i \log p_i$$

(Logs in base 2.)

Lyapunov exponent of (Φ, p)

$$\chi = \sum_{i \in \Lambda} p_i \int -\log |\varphi'_i(z)| \, d\mu(z)$$

Separated self-conformal sets: Sierpiński triangle; limit sets of Schottky semigroups generated by Möbius transformations; Julia sets of rational maps such as $z \mapsto z^2 + 3$.

Why? • Compute dimensions of overlapping self-conformal fractals • Obtain further developments in dimension rigidity

Dimension rigidity

For a self-conformal measure μ on $\mathbb{R}^d$,

$$\dim \mu = \min \left\{ \frac{H(p)}{\chi}, d \right\},$$

unless **obvious obstructions** occur.

Exact overlaps conjecture

For a **self-similar** measure μ associated to an IFS Φ on $\mathbb{R}$,

$$\dim \mu = \min \left\{ \frac{H(p)}{\chi}, 1 \right\},$$

whenever Φ **freely** generates a semigroup.

Progress

- Self-similar: Hochman ('14, '17), Shmerkin ('19), Varjú ('19), Rapaport ('22), D.-J. Feng–F. ('25), M. Wu ('25+), ...
- Projectively nonlinear: Hochman–Solomyak ('17), Rapaport–Ren ('25+).
- Real-analytic**: Rapaport ('24+).

What? We prove the expected dimension equality in $\mathbb{C}$ under a mild separation condition and essentially necessary assumptions.

Main theorem Let μ be the self-conformal measure associated to a holomorphic IFS Φ on a bounded domain $\Omega \subset \mathbb{C}$ and a positive probability vector p . Suppose

- Φ does not preserve any point or analytic curve that is closed in Ω .
- Φ is not holomorphically conjugate to a homothetic IFS.
- Φ is exponentially separated.

Then

$$\dim \mu = \min \left\{ \frac{H(p)}{\chi}, 2 \right\},$$

where $H(p)$ and χ denote entropy and Lyapunov exponent, respectively.

Definition

- Φ is *exponentially separated* if there exists $c > 0$ such that

$$\|\varphi_u - \varphi_v\|_{\Omega} > c^n,$$

for **infinitely** many $n \in \mathbb{N}$ and $u, v \in \Lambda^n$ with $u \neq v$, where

$$\|\varphi\|_{\Omega} = \sup_{z \in \Omega} |\varphi(z)| \quad \text{and} \quad \varphi_{i_1 \dots i_n} = \varphi_{i_1} \circ \dots \circ \varphi_{i_n}.$$

- Φ is *holomorphically conjugate to a homothetic IFS* if there exists a biholomorphic map $h: \Omega \rightarrow \mathbb{C}$ such that $\{h \circ \varphi \circ h^{-1}\}_{\varphi \in \Phi}$ consists of **homotheties**.

Remark

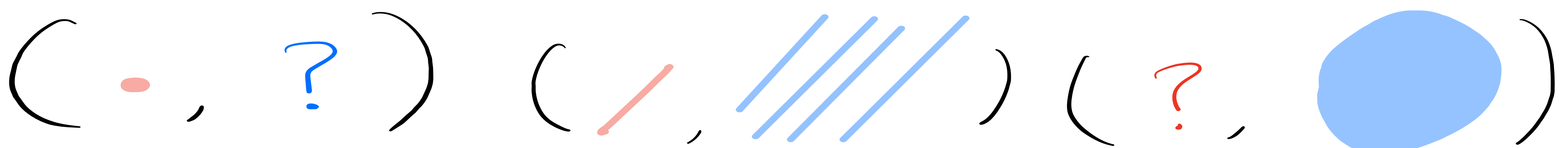
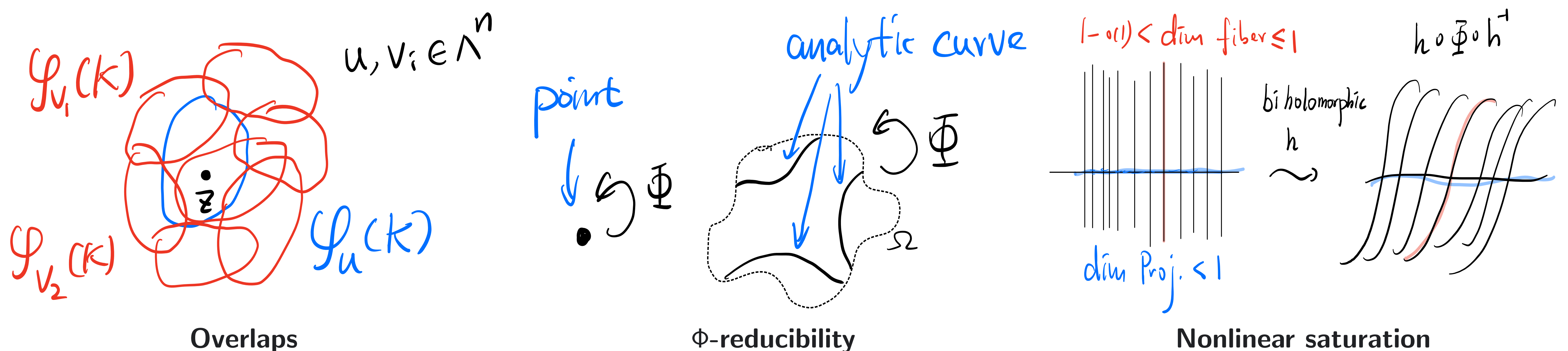
- ①** is **necessary** by Φ -*reducibility*.
- ②** is **essentially necessary** by *nonlinear saturation*.
- ③** is widely conjectured to be relaxed by Φ **freely** generating a semigroup.
- Bárány–Kolossváy–Troscheit ('25+) give **explicit** conditions for exponential separation in the **real-analytic** case.

Corollary

- The analogous dimension result for overlapping self-conformal **sets**.
- ①** and **②** $\Leftarrow$ the maps in Φ do not share a common fixed point, and there exists $u \in \bigcup_{n \in \mathbb{N}} \Lambda^n$ such that $\varphi'_u(z_u) \notin \mathbb{R}$, where $\varphi_u(z_u) = z_u$.
- Expected dimension equalities hold in a nice one-parameter family of holomorphic IFSs, outside a **zero-dimensional** exceptional set (+ Solomyak–Takahashi '22).

How? About the proof

- Taylor approximation technique** of Rapaport ('24+) + development of Hochman's argument ('14) $\rightarrow$ **entropy increase** under **polynomial** convolution.
- Linearization method** (see, e.g., Hochman–Solomyak ('17), Bárány–Hochman–Rapaport ('19)) $\rightarrow$ Hochman's inverse theorem for **translational** convolution in $\mathbb{R}^d$ ('17).
- Complex-analytic inputs**: *entropy porosity* $\leftarrow \mu(\gamma) = 0$ for analytic curve γ ; *entropy nonsaturation* $\leftarrow$ direction cocycle **not being a coboundary** (+ Rapaport–Ren ('25+)).



Obstructions to entropy increase for translational convolution $\xi * \theta$ of a pair of measures (ξ, θ) on $\mathbb{C}$

